

# Consistent couplings between spin-2 and spin-3 massless fields

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## Abstract

We solve the problem of constructing consistent first-order cross-interactions between spin-2 and spin-3 massless fields in flat spacetime of arbitrary dimension  $n > 3$  and in such a way that the deformed gauge algebra is non-Abelian. No assumptions are made on the number of derivatives involved in the Lagrangian, except that it should be finite. Together with locality, we also impose manifest Poincaré invariance, parity invariance and analyticity of the deformations in the coupling constants.

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# 1 Introduction

Although free higher-spin ( $s > 2$ ) gauge field theories are by now fairly well understood, the Fronsdal programme [1] which consists in introducing (non-Abelian) consistent interactions among these fields at the level of the action is still not achieved. Consistent nonlinear field equations for massless totally symmetric higher-spin fields in  $AdS_n$  background have been constructed [2], which represents a considerable achievement in higher-spin gauge field theory. Nonetheless, a corresponding action principle is lacking.

In this paper we adopt the metric-like formulation for higher-spin gauge fields [1, 3], consider collections of totally symmetric rank-2 and rank-3 gauge fields in flat space of arbitrary dimension  $n > 3$  and study the problem of introducing non-Abelian consistent cross-interactions among spin-2 and spin-3 massless fields. By “non-Abelian”, we mean that we focus on consistent deformations of the free theory such that the deformed gauge algebra becomes non-Abelian.

Demanding Poincaré invariance and locality, the non-Abelian self-interacting problems for collections of massless spin-2 and spin-3 fields, were respectively investigated in [4, 5] and [6, 7] by using the exhaustive BRST-BV cohomological method developed in [8, 9]. The works [4–7] revealed the existence of manifestly covariant cubic vertices which had not previously been written before. We refer to these works and [10, 11] for details and reviews on the self-interacting non-Abelian problem for spin-2 and spin-3 gauge fields in flat  $n$ -dimensional spacetime. The search for consistent higher-spin cubic vertices is a very important problem and other approaches exist. See e.g. [12] for a recent light-cone analysis in flat spacetime and more references on the problem of consistent higher-spin vertices, including Yang–Mills and gravitational couplings. See also [13] for a recent work concerning higher-spin vertices, including a discussion about the  $AdS_n$  background.

The Poincaré-invariant, local, non-Abelian consistent cross-interactions between spin-2 and spin-3 gauge fields in flat space remained to be analyzed in an exhaustive way and without any prejudice on the form of the interactions. In particular, we impose no upper limit on the number of derivatives appearing in the non-Abelian consistent vertex, apart that it should be finite in order that locality be preserved.

The advantage of the cohomological method [8, 9] which we use is that it enables one to classify and explicitly write down the consistent, nontrivial cubic vertices, without any other assumptions than locality and perturbative nature of the deformations. This method is also compatible with manifest Poincaré and gauge invariances, which is of great importance in the search for a possible geometrical interpretation of the higher-spin interactions. Since we have access to all the possible local, perturbative deformations of the gauge algebra and gauge transformations giving rise to nontrivial consistent cubic vertices, it can be hoped that the deformed gauge transformations

provide crucial information on a possible underlying nonlinear higher-spin geometry in flat space. Such a geometrical picture would in turn guide us toward a full nonlinear consistent Lagrangian.

Similarly to the self-interacting totally symmetric spin-2 and spin-3 cases [4–7], we first classify the possible first-order deformations of the gauge algebra and then determine which of these deformations give rise to nontrivial, consistent vertices. It turns out that only two parity-invariant algebra-deforming candidates satisfy this strong requirement. Interestingly enough, we find that, in order for the first candidate to induce a Poincaré-invariant nontrivial vertex, the (colored) spin-2 massless fields must react to the spin-3 field through a diffeomorphism-like transformation along the spin-3 gauge parameter, similarly to the way a spin-1 field reacts to a gravitational background via its Lie derivative along the diffeomorphism vector. Associated with the second algebra-deforming candidate is a gauge transformation of the spin-3 field along its own gauge parameter, but involving the linearized Riemann tensor for the spin-2 field. The first algebra-deforming candidate corresponds to the  $3 - 2 - 2$  covariant vertex mentioned in [14, 15], whereas the second algebra-deforming candidate gives rise to a nontrivial consistent  $2 - 3 - 3$  vertex which had previously not been written before, to our knowledge.

Our results therefore strengthen and complete those previously found in [14–16]. In particular, we recover in a simple way that both minimal and non-minimal couplings of spin-3 gauge fields to dynamical gravity in flat space are inconsistent [16].

In the work [14], consistent and covariant cubic couplings of the kind  $s_1 - s_2 - s_2$  were obtained, for the values of  $s_1$  and  $s_2$  indicated in Table 1. Of course, some of the vertices were already known

$\downarrow_{s_1} \rightarrow^{s_2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	×	×	×	×	×		
1	×	×	×	×	×		
2	×	×	×	×	×	×	
3	×	×	×	×	×	×	×
$n$	×						

Table 1:  $s_1 - s_2 - s_2$  covariant vertices obtained in [14].

before, like for example in the cases  $1 - 1 - 1$ ,  $2 - 2 - 2$  and  $2 - \frac{3}{2} - \frac{3}{2}$  corresponding to Yang–Mills, Einstein–Hilbert and ordinary supergravity theories. There is a class of cross-interactions  $s_1 - s_2 - s_2$  for which the cubic vertices could easily been written. This class corresponds to the “Bell–Robinson” line  $s_1 = 2s_2$  and below this line  $s_1 > 2s_2$  [15] (see [17] in the particular  $s_1 = 4 = 2s_2$  case). In the aforementioned region  $s_1 \geq 2s_2$ , the gauge algebra remains Abelian although the gauge transformations for the spin- $s_2$  field are deformed at first order in a coupling

constant. The reason is that the first-order deformation of the free spin- $s_2$  gauge transformations involve the spin- $s_2$  field only through its gauge-invariant Weinberg–de Wit–Freedman field-strength [3, 18]<sup>3</sup>. Although they do not lead to non-Abelian gauge algebras, it is interesting that the cubic interactions on and below the Bell–Robinson line (*i.e.* for  $s_1 \geq 2s_2$ ) have the form “spin- $s_1$  field times current  $J$ ” where  $J$  is quadratic in the spin- $s_2$  field-strength [15, 17] and is conserved on the spin- $s_2$  shell. Even more interestingly, these currents can be obtained from some global invariances of the free theory by a Noether-like procedure, provided the constant parameters associated with these rigid symmetries be replaced by the gauge parameters of the spin- $s_1$  field (also internal indices must be treated appropriately) [15, 17].

In the present paper, we re-derive the non-Abelian  $3 - 2 - 2$  cubic vertex mentioned in [14], show that it is inconsistent when pushed up to second order in the deformation parameter and obtain a consistent  $2 - 3 - 3$  vertex which passes the second-order consistency test where the former Berends–Burgers–van Dam vertex fails. Moreover, at the level of the Jacobi identity at second order in the coupling constant, we show that the latter  $2 - 3 - 3$  covariant vertex is compatible with the spin-3 self-coupling written in [20]. Also, even though the  $3 - 2 - 2$  vertex stands above the Bell–Robinson line  $s_1 = 2s_2$ , we show that it can be seen as partially resulting from the gauging of the global symmetries discussed in [15]. Of course, this vertex truly deforms the gauge algebra and consequently the coupling cannot be written in the simple form outlined before. The gauge transformations for both spin-2 and spin-3 fields are nontrivially deformed.

The plan of the paper is as follows. In the next section we recall some basic facts on the free spin-2 and spin-3 gauge theories and on the BRST-antifield formalism used throughout the text. Section 3 gathers together some BRST-cohomological results that are needed. Section 4 contains most of our computations and results concerning the first-order consistent couplings between spin-2 and spin-3 massless fields. In Section 5 we present the constraints that are imposed on the first-order deformations by second-order consistency conditions. We also discuss the links between the first-order gauge transformations presented in Section 4 and some results of [15, 17]. Finally, our conclusions and perspectives are given in Section 6. The first appendix contains a technical BRST-cohomological result. The complete expressions for the first-order vertices are displayed in the second appendix.

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<sup>3</sup>Note that one can write down higher-derivative Born–Infeld-like consistent cubic interactions involving only gauge-invariant field-strength tensors [19]. However, these interactions deform neither the gauge algebra nor the gauge transformations. They are not considered in the present work but are accounted for in the powerful light-cone approach presented in [12].

## 2 Free theory and BRST settings

### 2.1 Free Theory

The action for a collection  $\{h_{\mu\nu}^a\}$  of  $M$  *non-interacting*, massless spin-2 fields in spacetime dimension  $n$  ( $\mu, \nu = 0, \dots, n-1$ ) is (equivalent to) the sum of  $M$  separate Pauli-Fierz actions, namely

$$S_0^2[h_{\mu\nu}^a] = \sum_{a=1}^M \int \delta_{ab} \left[ -\frac{1}{2} (\partial_\mu h^a_{\nu\rho}) (\partial^\mu h^{b\nu\rho}) + (\partial_\mu h^{a\mu}_{\nu}) (\partial_\rho h^{b\rho\nu}) \right. \\ \left. - (\partial_\nu h^{a\mu}_{\mu}) (\partial_\rho h^{b\rho\nu}) + \frac{1}{2} (\partial_\mu h^{a\nu}_{\nu}) (\partial^\mu h^{b\rho}_{\rho}) \right] d^n x, \quad n > 2. \quad (2.1)$$

The lower-case Latin indices are internal indices taking  $M$  values. They are raised and lowered with the Kronecker delta's  $\delta^{ab}$  and  $\delta_{ab}$ . The Greek indices are space-time indices taking  $n$  values, which are lowered (resp. raised) with the “mostly plus” Minkowski metric  $\eta_{\mu\nu}$  (resp.  $\eta^{\mu\nu}$ ). The action (2.1) is invariant under the following linear gauge transformations,

$$\delta_\epsilon h_{\mu\nu}^a = \partial_\mu \epsilon_\nu^a + \partial_\nu \epsilon_\mu^a \quad (2.2)$$

where the  $\epsilon_\nu^a$  are  $n \times M$  arbitrary, independent functions. These transformations are Abelian and irreducible. The equations of motion are

$$\frac{\delta S_0^2}{\delta h_{\mu\nu}^a} = -2H_a^{\mu\nu} = 0$$

where  $H_{\mu\nu}^a$  is the linearized Einstein tensor,

$$H_{\mu\nu}^a = K_{\mu\nu}^a - \frac{1}{2} K^a \eta_{\mu\nu}.$$

Here,  $K_{\alpha\beta\mu\nu}^a$  is the linearized Riemann tensor,

$$K_{\alpha\beta\mu\nu}^a = -\frac{1}{2} (\partial_{\alpha\mu} h_{\beta\nu}^a + \partial_{\beta\nu} h_{\alpha\mu}^a - \partial_{\alpha\nu} h_{\beta\mu}^a - \partial_{\beta\mu} h_{\alpha\nu}^a),$$

$K_{\mu\nu}^a$  is the linearized Ricci tensor,

$$K_{\mu\nu}^a = K^{a\alpha}_{\mu\alpha\nu} = -\frac{1}{2} (\square h_{\mu\nu}^a + \dots),$$

and  $K^a$  is the linearized scalar curvature,  $K^a = \eta^{\mu\nu} K_{\mu\nu}^a$ . The Noether identities expressing the invariance of the free action (2.1) under (2.2) are

$$\partial_\nu H^{a\mu\nu} = 0 \quad (2.3)$$

(linearized Bianchi identities). The gauge symmetry removes unwanted unphysical states.

The local action for a collection  $\{h_{\mu\nu\rho}^A\}$  of  $N$  non-interacting totally symmetric massless spin-3 gauge fields in flat spacetime is [1]

$$S_0^3[h_{\mu\nu\rho}^A] = \sum_{A=1}^N \int \delta_{AB} \left[ -\frac{1}{2} \partial_\sigma h_{\mu\nu\rho}^A \partial^\sigma h^{B\mu\nu\rho} + \frac{3}{2} \partial^\mu h_{\mu\rho\sigma}^A \partial_\nu h^{B\nu\rho\sigma} + \right. \\ \left. \frac{3}{2} \partial_\mu h_\nu^A \partial^\mu h^{B\nu} + \frac{3}{4} \partial_\mu h^{A\mu} \partial_\nu h^{B\nu} - 3 \partial_\mu h_\nu^A \partial_\rho h^{B\rho\mu\nu} \right] d^n x, \quad (2.4)$$

where  $h_\mu^A = \eta^{\nu\rho} h_{\mu\nu\rho}^A$ . The upper-case Latin indices are internal indices taking  $N$  values. They are raised and lowered with the Kronecker delta's  $\delta^{AB}$  and  $\delta_{AB}$ . The action (2.4) is invariant under the gauge transformations

$$\delta_\lambda h_{\mu\nu\rho}^A = 3 \partial_{(\mu} \lambda_{\nu\rho)}^A, \quad \eta^{\mu\nu} \lambda_{\mu\nu}^A = 0, \quad (2.5)$$

where the gauge parameters  $\lambda_{\nu\rho}^A$  are symmetric and traceless. Curved (resp. square) brackets on spacetime indices denote strength-one complete symmetrization (resp. antisymmetrization) of the indices. The gauge transformations (2.5) are Abelian and irreducible. The field equations read

$$\frac{\delta S_0^3}{\delta h_{\mu\nu\rho}^A} = G_A^{\mu\nu\rho} = 0, \quad (2.6)$$

where

$$G_{\mu\nu\rho}^A = F_{\mu\nu\rho}^A - \frac{3}{2} \eta_{(\mu\nu} F_{\rho)}^A \quad (2.7)$$

is the ‘‘Einstein’’ tensor and  $F_{\mu\nu\rho}^A$  the Fronsdal (or ‘‘Ricci’’) tensor

$$F_{\mu\nu\rho}^A = \square h_{\mu\nu\rho}^A - 3 \partial^\sigma \partial_{(\mu} h_{\nu\rho)\sigma}^A + 3 \partial_{(\mu} \partial_\nu h_{\rho)}^A. \quad (2.8)$$

The Fronsdal tensor is gauge invariant thanks to the tracelessness of the gauge parameters. Because we have  $\delta_\lambda S_0[h_{\mu\nu\rho}^A] = 0$  for the gauge transformations (2.5), the Einstein tensor  $G_{\mu\nu\rho}^A$  satisfies the Noether identities

$$\partial^\rho G_{\mu\nu\rho}^A - \frac{1}{n} \eta_{\mu\nu} \partial^\rho G_\rho^A = 0 \quad (G_\rho^A = \eta^{\mu\nu} G_{\mu\nu\rho}^A) \quad (2.9)$$

related to the symmetries of the gauge parameters  $\lambda_{\mu\nu}^A$ ; in other words, the l.h.s. of (2.9) is symmetric and traceless.

An important object is the Weinberg–de Wit–Freedman (or ‘‘Riemann’’) spin-3 tensor [3, 18, 25]

$$K_{\alpha\mu|\beta\nu|\gamma\rho}^A = 8 \partial_{[\gamma} \partial_{[\beta} \partial_{\alpha} h_{\mu]\nu|\rho]}^A$$

which is antisymmetric in  $\alpha\mu$ ,  $\beta\nu$ ,  $\gamma\rho$  and invariant under gauge transformations (2.5), where the gauge parameters  $\lambda_{\mu\nu}^A$  are however *not* necessarily traceless. Its importance, apart from gauge invariance with unconstrained gauge parameters, stems from the fact that the field equations (2.6) are equivalent to the following equations

$$\eta^{\alpha\beta} K_{\alpha\mu|\beta\nu|\gamma\rho}^A = 0,$$

after a partial gauge fixing. This was proved in the work [23] by combining various former results [24–26]. See [27] for more details and for the arbitrary mixed-symmetry case.

## 2.2 BRST spectrum and differential

According to the general rules of the BRST-antifield formalism, the field spectrum consists of the fields  $\{h_{\mu\nu}^a, h_{\mu\nu\rho}^A\}$ , the ghosts  $\{C_\mu^a, C_{\mu\nu}^A\}$ , the antifields  $\{h_a^{*\mu\nu}, h_A^{*\mu\nu\rho}\}$  and the ghost antifields  $\{C_a^{*\mu}, C_A^{*\mu\nu}\}$ . The set of fields and ghosts will sometimes be collectively denoted by  $\Phi^I$ , whereas the associated set of antifields will be denoted by  $\Phi_I^*$ .

The BRST differential  $s$  of the free theory  $S_0[h^a, h^A] = S_0^2[h_{\mu\nu}^a] + S_0^3[h_{\mu\nu\rho}^A]$  is generated by the functional

$$W_0 = S_0[h^a, h^A] + \int (2 h_a^{*\alpha\beta} \partial_\alpha C_\beta^a + 3 h_A^{*\mu\nu\rho} \partial_\mu C_{\nu\rho}^A) d^n x.$$

More precisely,  $W_0$  is the generator of the BRST differential  $s$  of the free theory through

$$sA = (W_0, A),$$

where the antibracket  $(\ , \ )$  is defined by

$$(A, B) = \frac{\delta^R A}{\delta \Phi^I} \frac{\delta^L B}{\delta \Phi_I^*} - \frac{\delta^R A}{\delta \Phi_I^*} \frac{\delta^L B}{\delta \Phi^I},$$

using the condensed de Witt notation in which a summation over a repeated index also implies an integration over spacetime variables.

The functional  $W_0$  is a solution of the *master equation*

$$(W_0, W_0) = 0.$$

In the theory at hand, the BRST-differential  $s$  decomposes into  $s = \gamma + \delta$ . The first piece  $\gamma$ , the differential along the gauge orbits, is associated with the  $\mathbb{N}$ -grading called pureghost number (*puregh*) and increases it by one unit, whereas the Koszul-Tate differential  $\delta$  decreases the grading called antighost number (*antigh*) by one unit. The differential  $s$  increases the  $\mathbb{Z}$ -grading called

ghost number ( $gh$ ) by one unit. Furthermore, the ghost, antighost and pureghost gradings are not independent. We have the relation

$$gh = puregh - antigh.$$

The pureghost number, antighost number, ghost number and grassmannian parity of the various fields are displayed in Table 2. The action of the differentials  $\delta$  and  $\gamma$  gives zero on all the fields of

$Z$	$puregh(Z)$	$antigh(Z)$	$gh(Z)$	parity (mod 2)
$\{h_{\mu\nu}^a, h_{\mu\nu\rho}^A\}$	0	0	0	0
$\{C_\mu^a, C_{\mu\nu}^A\}$	1	0	1	1
$\{h_a^{*\mu\nu}, h_A^{*\mu\nu\rho}\}$	0	1	-1	1
$\{C_a^{*\mu}, C_A^{*\mu\nu}\}$	0	2	-2	0

Table 2: *pureghost number, antighost number, ghost number and parity of the (anti)fields.*

the formalism except in the few following cases:

$$\begin{aligned}\delta h_A^{*\mu\nu\rho} &= G_A^{\mu\nu\rho}, & \delta h_a^{*\mu\nu} &= -2H_a^{\mu\nu}, \\ \delta C_A^{*\mu\nu} &= -3(\partial_\rho h_A^{*\mu\nu\rho} - \frac{1}{n}\eta^{\mu\nu}\partial_\rho h_A^{*\rho}), & \delta C_a^{*\mu} &= -2\partial_\nu h_a^{*\nu\mu}, \\ \gamma h_{\mu\nu\rho}^A &= 3\partial_{(\mu} C_{\nu\rho)}^A, & \gamma h_{\mu\nu}^a &= 2\partial_{(\mu} C_{\nu)}^a.\end{aligned}$$

More details about the antifield formalism for spin-2 and spin-3 can be found in [4, 6].

### 2.3 BRST deformations

As shown in [8], the Noether procedure can be reformulated within a BRST-cohomological framework. Any consistent deformation of the gauge theory corresponds to a solution

$$W = W_0 + gW_1 + g^2W_2 + \mathcal{O}(g^3)$$

of the deformed master equation  $(W, W) = 0$ . Consequently, the first-order nontrivial consistent local deformations  $W_1 = \int a^{n,0}$  are in one-to-one correspondence with elements of the cohomology  $H^{n,0}(s|d)$  of the zeroth order BRST differential  $s = (W_0, \cdot)$  modulo the total derivative  $d$ , in maximum form-degree  $n$  and in ghost number 0. That is, one must compute the general solution of the cocycle condition

$$sa^{n,0} + db^{n-1,1} = 0, \tag{2.10}$$

where  $a^{n,0}$  is a top-form of ghost number zero and  $b^{n-1,1}$  a  $(n-1)$ -form of ghost number one, with the understanding that two solutions of (2.10) that differ by a trivial solution should be identified

$$a^{n,0} \sim a^{n,0} + sp^{n,-1} + dq^{n-1,0},$$



as they define the same interactions up to field redefinitions. The cocycles and coboundaries  $a, b, p, q, \dots$  are local forms of the field variables, including ghosts and antifields (those are forms acting on the jet space  $J_k$ , the vectorial space generated by the fields and a finite number  $k$  of their derivatives).

The corresponding second-order interactions  $W_2$  must satisfy the consistency condition

$$sW_2 = -\frac{1}{2}(W_1, W_1). \quad (2.11)$$

This condition is controlled by the local BRST cohomology group  $H^{n,1}(s|d)$ .

### 3 Cohomological results

#### 3.1 Cohomology of $\gamma$

In the context of local free theories in Minkowski space for massless spin- $s$  gauge fields represented by totally symmetric (and double traceless when  $s > 3$ ) rank  $s$  tensors, the groups  $H^*(\gamma)$  have been calculated in [28]. When a sum of several such theories for different spins is considered, the cohomology is the direct product of the cohomologies of the different theories. We will prove it only in the case of a sum of spin-2 and spin-3 Fronsdaal theories, but the proof can straightforwardly be extended.

**Proposition 1.** *The cohomology of  $\gamma$  is isomorphic to the space of functions depending on*

- the antifields  $\{h_a^{*\mu\nu}, h_A^{*\mu\nu\rho}, C_a^{*\mu}, C_A^{*\mu\nu}\}$  and their derivatives, denoted by  $[\Phi_I^*]$ ,
- the curvatures and their derivatives  $[K_{\alpha\mu|\beta\nu}^a], [K_{\alpha\mu|\beta\nu|\gamma\rho}^A]$ ,
- the symmetrized derivatives  $\partial_{(\sigma_1} \dots \partial_{\sigma_k} F_{\mu\nu\rho)}^A$  of the Fronsdaal tensor,
- the ghosts  $C_\mu^a$  and their antisymmetrized first-order derivatives  $\partial_{[\mu} C_{\nu]}^a$ , the ghosts  $C_{\mu\nu}^A$  and the traceless parts of  $\partial_{[\mu} C_{\nu]\rho}^A$  and<sup>4</sup>  $\partial_{[\mu} C_{\nu][\rho,\sigma]}^A$ .

Thus, identifying with zero any  $\gamma$ -exact term in  $H(\gamma)$ , we have

$$\gamma f = 0$$

if and only if

$$f = f \left( [\Phi_I^*], [K_{\alpha\mu|\beta\nu|\gamma\rho}^A], [K_{\alpha\mu\beta\nu}^a], \{F_{\mu\nu\rho}^A\}, C_\nu^a, \partial_{[\mu} C_{\nu]}^a, C_{\mu\nu}^A, \hat{T}_{\alpha\mu|\nu}^A, \hat{U}_{\alpha\mu|\beta\nu}^A \right)$$

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<sup>4</sup>A coma denotes a partial derivative, e.g.  $\Phi_{,\mu}^I = \partial_\mu \Phi^I$ .

where  $\{F_{\mu\nu\rho}^A\}$  stands for the completely symmetrized derivatives  $\partial_{(\sigma_1}\dots\partial_{\sigma_k}F_{\mu\nu\rho)}^A$  of the Fronsdal tensor, while  $\widehat{T}_{\rho\mu|\nu}^A$  denotes the traceless part of  $T_{\rho\mu|\nu}^A = \partial_{[\rho}C_{\mu]\nu}^A$  and  $\widehat{U}_{\rho\mu|\sigma\nu}^A$  the traceless part of  $U_{\rho\mu|\sigma\nu}^A = \partial_{[\rho}C_{\mu]|\nu,\sigma]}^A$ .

Let  $\{\omega^I\}$  be a basis of the space of polynomials in the  $C_\mu^a$ ,  $\partial_{[\mu}C_{\nu]}^a$ ,  $C_{\mu\nu}^A$ ,  $\widehat{T}_{\alpha\mu|\nu}^A$  and  $\widehat{U}_{\alpha\mu|\beta\nu}^A$  (since these variables anticommute, this space is finite-dimensional). If a local form  $a$  is  $\gamma$ -closed, we have

$$\gamma a = 0 \Rightarrow a = \alpha_J([\Phi_I^*], [K^a], [K^A], \{F^A\}) \omega^J(C_\mu^a, \partial_{[\mu}C_{\nu]}^a, C_{\mu\nu}^A, \widehat{T}_{\alpha\mu|\nu}^A, \widehat{U}_{\alpha\mu|\beta\nu}^A) + \gamma b. \quad (3.12)$$

If  $a$  has a fixed, finite ghost number, then  $a$  can only contain a finite number of antifields. Moreover, since the *local* form  $a$  possesses a finite number of derivatives, we find that the  $\alpha_J$  are polynomials. Such a polynomial  $\alpha_J([\Phi_I^*], [K^a], [K^A], \{F^A\})$  will be called an *invariant polynomial*. The proof of Proposition 1 is given in appendix A.

**Remark:** Because of the Damour-Deser identity [25]

$$\eta^{\alpha\beta} K_{\alpha\mu|\beta\nu|\gamma\rho}^A = 2 \partial_{[\gamma} F_{\rho]\mu\nu}^A,$$

the derivatives of the Fronsdal tensor  $F^A$  are not all independent of the curvature tensor  $K^A$ . This is why, in Proposition 1, the completely symmetrized derivatives of  $F^A$  appear, together with all the derivatives of the curvature  $K^A$ . However, from now on, we will assume that every time the trace  $\eta^{\alpha\beta} K_{\alpha\mu|\beta\nu|\gamma\rho}^A$  appears, we substitute  $2\partial_{[\gamma} F_{\rho]\mu\nu}^A$  for it. With this convention, we can write  $\alpha_J([\Phi_I^*], [K^a], [K^A], [F^A])$  instead of the inconvenient notation  $\alpha_J([\Phi^{i*}], [K^a], [K^A], \{F\})$ .

### 3.2 Invariant Poincaré lemma

We shall need several standard results on the cohomology of  $d$  in the space of invariant polynomials.

**Proposition 2.** *In form degree less than  $n$  and in antifield number strictly greater than 0, the cohomology of  $d$  is trivial in the space of invariant polynomials. That is to say, if  $\alpha$  is an invariant polynomial, the equation  $d\alpha = 0$  with  $\text{antigh}(\alpha) > 0$  implies  $\alpha = d\beta$  where  $\beta$  is also an invariant polynomial.*

The latter property is rather generic for gauge theories (see e.g. Ref. [4] for a proof), as well as the following:

**Proposition 3.** *If  $a$  has strictly positive antifield number, then the equation  $\gamma a + db = 0$  is equivalent, up to trivial redefinitions, to  $\gamma a = 0$ . More precisely, one can always add  $d$ -exact terms to  $a$  and get a cocycle  $a' = a + dc$  of  $\gamma$ , such that  $\gamma a' = 0$ .*

**Proof:** See e.g. [6].

### 3.3 Cohomology of $\delta$ modulo $d$ : $H_k^n(\delta|d)$

In this section, we review the local Koszul-Tate cohomology groups in top form-degree and antighost numbers  $k \geq 2$ . The group  $H_1^n(\delta|d)$  describes the infinitely many conserved currents and will not be studied here.

Let us first recall a general result (Theorem 9.1 in [29]).

**Proposition 4.** *For a linear gauge theory of reducibility order  $r$ ,*

$$H_p^n(\delta|d) = 0 \text{ for } p > r + 2.$$

Since the theory at hand has no reducibility, we are left with the computation of  $H_2^n(\delta|d)$ . The cohomology  $H_2^n(\delta|d)$  is given by the following theorem.

**Proposition 5.** *A complete set of representatives of  $H_2^n(\delta|d)$  is given by the antifields  $\{C_a^{*\mu}, C_A^{*\mu\nu}\}$ , up to explicitly  $x$ -dependent terms. In detail,*

$$\begin{cases} \delta a_2^n + db_1^{n-1} = 0 \\ a_2^n \sim a_2^n + \delta c_3^n + dc_2^{n-1} \end{cases} \iff a_2^n = [\lambda_\mu^a(x)C_a^{*\mu} + L_{\mu\nu}^A(x)C_A^{*\mu\nu}] d^n x + \delta b_3^n + db_2^{n-1}$$

where  $\lambda_\mu^a(x) = a_\mu^a + \dots$  is a degree-1 polynomial in  $x^\nu$  and  $L_{\mu\nu}^A(x) = \lambda_{\mu\nu}^A + \dots$  is a degree-2 polynomial.

The coefficients of these polynomials have definite symmetry properties that we will not recall here. The complete analysis can be found in [4, 6]. See also [30, 31]. From the requirement of Poincaré invariance, explicit dependence in the coordinates is forbidden and we will only consider the constant terms  $a_\mu^a$  and  $\lambda_{\mu\nu}^A$  in the expansions of  $\lambda_\mu^a(x)$  and  $L_{\mu\nu}^A(x)$ .

The most general  $n$ -form in *antigh* 2 is  $a = (f_\mu^a C_a^{*\mu} + \hat{f}_{\mu\nu}^A C_A^{*\mu\nu}) d^n x + \Phi + \delta b + dc$ , where  $\Phi$  is quadratic in the *antigh*-1 antifields. If one applies  $\delta$ , the  $\delta$ -exact term vanishes and  $\delta\Phi + df \approx 0$ . So, if  $a \in H_2^n(\delta|d)$ , the weak equality  $(-2f_\mu^a \partial_\nu h^{*\mu\nu} - 3\hat{f}_{\mu\nu}^A \partial_\rho h_A^{*\mu\nu\rho}) d^n x \approx dv$  is obtained. Finally, by applying variational derivatives with respect to  $h_a^{*\mu\nu}$  and  $h_A^{*\mu\nu\rho}$ , the two weak equalities  $\partial_{(\mu} f_{\nu)}^a \approx 0$  and  $\partial_{(\mu} \hat{f}_{\nu\rho)}^A \approx 0$  are obtained. These are both on-shell Killing equations for the individual spin-2 and spin-3 cases. Each equation of the type  $\partial_{(\mu_1} \hat{f}_{\mu_2 \dots \mu_s)} \approx 0$  provides  $H_2^n(\delta|d)$  for the pure spin- $s$  case, the solutions of which have been given in Ref. [28] (see also [30]). This is because those solutions are  $\delta$ -closed modulo  $d$  and because  $\Phi$  obeying  $\delta\Phi + dc = 0$  is a trivial cocycle [32]. The spin-2 case under consideration was already written in Ref. [4] and the spin-3 case was written in [6, 31]. In any mixed case, the different equations for the different spins will have to be satisfied and  $H_2^n(\delta|d)$  is then the direct sum of the individual cases.

We have studied above the cohomology of  $\delta$  modulo  $d$  in the space of arbitrary local functions of the fields, the antifields, and their derivatives. One can also study  $H_k^n(\delta|d)$  in the space of invariant

polynomials in these variables. The above theorems remain unchanged in this space, *i.e.* we have the

**Proposition 6.** *The invariant cohomology  $H_k^n(\delta|d, H_0(\gamma))$  is trivial in antighost number  $k > 2$ . In antighost number  $k = 2$ , we have the isomorphism  $H_2^n(\delta|d, H_0(\gamma)) \cong H_2^n(\delta|d)$ .*

This very nontrivial property is crucial for the computation of  $H^{n,0}(s|d)$ . It has been proved for the spin-2 case in [4] and for the spin-3 case in [6] for  $n > 3$ , and [7] for  $n = 3$ . In the mixed spin-2–spin-3 case, the proof goes along the same lines. It has to be checked that, in a coboundary in form degree  $n$  :  $a_k = \delta b_{k+1} + \partial_\mu j_k^\mu$  (in dual notation), if  $a_k$  is invariant, then  $b_{k+1}$  and  $j_k^\mu$  can be chosen as being invariant. This is done by reconstructing  $a_k$  from its variational derivatives with respect to the different fields and antifields (see [4], Lemma A.2 and [6], section 4.6.2). The considerations made for the spin-2 and spin-3 derivatives hold independently here, and  $a_k$  can be reconstructed in an invariant way with no further problems.

### 3.4 Definition of the $D$ -degree

**Definition (differential  $D$ ):** The action of the differential  $D$  on the fields, the antifields and all their derivatives is the same as the action of the total derivative  $d$ , but its action on the ghosts is given by:

$$\begin{aligned} DC_{\mu\nu}^A &= \frac{4}{3} dx^\alpha \hat{T}_{\alpha(\mu|\nu)}^A, \\ D\hat{T}_{\mu\alpha|\beta}^A &= dx^\rho \hat{U}_{\mu\alpha|\rho\beta}^A, \\ D(\partial_{\rho_1 \dots \rho_t} C_{\mu\nu}^A) &= 0 \text{ if } t \geq 2, \\ DC_\mu^a &= dx^\nu \partial_{[\nu} C_{\mu]}^a, \\ D(\partial_{\rho_1 \dots \rho_v} C_\mu^a) &= 0 \text{ if } v \geq 1. \end{aligned}$$

The above definitions follow from

$$\begin{aligned} \partial_\mu C_\nu^a &= \frac{1}{2} (\gamma h_{\mu\nu}^a) + \partial_{[\mu} C_{\nu]}^a, \\ \partial_\alpha C_{\mu\nu}^A &= \frac{1}{3} (\gamma h_{\alpha\mu\nu}^A) + \frac{4}{3} T_{\alpha(\mu|\nu)}^A, \\ \partial_\rho T_{\mu\alpha|\beta}^A &= -\frac{1}{2} \gamma (\partial_{[\alpha} h_{\mu]}^A)_{\beta\rho} + U_{\mu\alpha|\rho\beta}^A, \\ \partial_\rho U_{\mu\alpha|\nu\beta}^A &= \frac{1}{3} \gamma (\partial_{[\mu} h_{\alpha]}^A)_{\rho[\beta,\nu]}. \end{aligned}$$

The operator  $D$  thus coincides with  $d$  up to  $\gamma$ -exact terms.

It follows from the definitions that  $D\omega^J = A^J_I \omega^I$  for some constant matrix  $A^J_I$  that involves  $dx^\mu$  only. It is also convenient to introduce a new grading.

**Definition ( $D$ -degree):** The number of  $\hat{T}_{\alpha\mu|\nu}^A$ 's and  $\partial_{[\mu}C_{\nu]}^a$ 's plus twice the number of  $\hat{U}_{\alpha\mu|\beta\nu}^A$ 's is called the  $D$ -degree. It is bounded because there is a finite number of  $\partial_{[\mu}C_{\nu]}^a$ 's,  $\hat{T}_{\alpha\mu|\nu}^A$ 's and  $\hat{U}_{\alpha\mu|\beta\nu}^A$ 's which are anticommuting. The operator  $D$  splits as the sum of an operator  $D_1$  that raises the  $D$ -degree by one unit and an operator  $D_0$  that leaves it unchanged.  $D_0$  has the same action as  $d$  on the fields, the antifields and all their derivatives, and gives 0 when acting on the ghosts.  $D_1$  gives 0 when acting on all the variables but the ghosts on which it reproduces the action of  $D$ .

## 4 First-order consistent deformations

As recalled in Section 2.3, nontrivial consistent interactions are in one-to-one correspondence with elements of  $H^{n,0}(s|d)$ , *i.e.* solutions  $a$  of the equation

$$sa + db = 0, \quad (4.13)$$

with form-degree  $n$  and ghost number zero, modulo the equivalence relation

$$a \sim a + sp + dq.$$

Quite generally, one can expand  $a$  according to the antighost number, as

$$a = a_0 + a_1 + a_2 + \dots a_k, \quad (4.14)$$

where  $a_i$  has antighost number  $i$ . The expansion stops at some finite value of the antighost number by locality, as was proved in [32].

Let us recall [9] the meaning of the various components of  $a$  in this expansion. The antifield-independent piece  $a_0$  is the deformation of the Lagrangian;  $a_1$ , which is linear in the antifields  $h^*$ , contains the information about the deformation of the gauge symmetries, given by the coefficients of  $h^*$ ;  $a_2$  contains the information about the deformation of the gauge algebra (the term  $C^*CC$  gives the deformation of the structure functions appearing in the commutator of two gauge transformations, while the term  $h^*h^*CC$  gives the on-shell closure terms); and the  $a_k$  ( $k > 2$ ) give the informations about the deformation of the higher order structure functions and the reducibility conditions.

### 4.1 Equations

In fact, using the previous cohomological theorems and standard reasonings (see e.g. [4]), one can remove all components of  $a$  with antifield number greater than 2. The key point is that the invariant characteristic cohomology  $H_k^{n,inv}(\delta|d)$  controls the obstructions to the removal of the term  $a_k$  from  $a$  and that all  $H_k^{n,inv}(\delta|d)$  vanish for  $k > 2$  by Proposition 4 and Proposition 6.

Let us now decompose the cocycle condition (4.13) according to the antighost number. If  $a = a_0 + a_1 + a_2$ , then  $b$  can be assumed to stop at antigh 1 thanks to Proposition 3. Using the fact that  $s = \delta + \gamma$ , we obtain:

$$\gamma a_2 = 0, \quad (4.15)$$

$$\delta a_2 + \gamma a_1 + db_1 = 0, \quad (4.16)$$

$$\delta a_1 + \gamma a_0 + db_0 = 0. \quad (4.17)$$

The first equation clearly means that  $[a_2] \in H^2(\gamma) \Leftrightarrow a_2 = \alpha_J \omega^J + \gamma c_2$  as in Equation (3.12). Applying  $\gamma$  to Equation (4.16),  $d\gamma b_1 = 0$  is obtained. Thanks to the Poincaré lemma and Proposition 3, we see that  $b_1$  can be taken in  $H^2(\gamma)$  too :  $b_1 = \beta_J \omega^J$ . The second equation becomes

$$(\delta \alpha_J) \omega^J + \gamma a_1 + d\beta_J \omega^J + \beta_J d\omega^J = 0.$$

Let us now introduce the differential  $D$  defined in Section 3.4, we obtain

$$(\delta \alpha_J + d\beta_J + \beta_I A_J^I) \omega^J = \gamma(\dots) = 0.$$

This is because the left-hand side is strictly non  $\gamma$ -exact. Let us label the ghosts more precisely  $\omega^{J_i}$  where  $i$  is the  $D$ -degree. Then, as  $D$  raises by 1 the  $D$ -degree, the only non zero components of the matrix  $A$  are  $A_{J_{i+1}}^{J_i}$  and the last equation decomposes into:

$$\begin{aligned} \delta \alpha_{J_0} + d\beta_{J_0} &= 0, \\ \forall i > 0 : \delta \alpha_{J_i} + d\beta_{J_i} + \beta_{J_{i-1}} A_{J_i}^{J_{i-1}} &= 0. \end{aligned}$$

The first equation means that

$$\alpha_{J_0} \in H_2^n(\delta|d, H_0(\gamma)) \Rightarrow \alpha_{J_0} = [\lambda_{J_0 a \mu} C^{*a\mu} + \lambda_{J_0 A \mu \nu} C^{*A\mu\nu}] d^n x,$$

thanks to Proposition 5 and Proposition 6. The  $\lambda$ 's are constants, because of the Poincaré invariance. We obtain

$$\beta_{J_0} = \frac{-1}{(n-1)!} [2\lambda_{J_0 a \mu} h^{*a\mu\alpha} + 3\lambda_{J_0 A \mu \nu} C^{*A\mu\nu\alpha}] \varepsilon_{\alpha\mu_1 \dots \mu_{n-1}} dx^{\mu_1} \dots dx^{\mu_{n-1}}.$$

Thus,  $\beta_{J_0} A_{J_1}^{J_0}$  depends only on the underivated antifields, which cannot be  $\delta$ -exact modulo  $d$  unless they vanish, because a  $\delta$ -exact term depends on the derivatives of the antifields or on the equations of motion, and because of the Poincaré invariance. Thus,  $\delta \alpha_{J_1} + d\beta_{J_1} = 0$  and  $\beta_{J_0} A_{J_1}^{J_0} = 0$  independently. By applying the same reasoning recursively, the same decomposition appears to occur at every  $D$ -degree, so we finally obtain:

$$\begin{aligned} \forall i : \delta \alpha_{J_i} + d\beta_{J_{i-1}} &= 0 \Rightarrow \alpha_{J_i} \in H_{2,inv}(\delta|d), \\ \forall i : \beta_{J_i} A_{J_{i+1}}^{J_i} &= 0. \end{aligned} \quad (4.18)$$

## 4.2 Classification of the gauge algebras

The first set of equations (4.18) provides a very limited number of candidates  $a_2$ . The different possible Lorentz-invariant terms have the form  $\alpha_{J_i} \omega^{J_i} = \lambda_{J_i \dots} C^{* \dots} \omega^{J_i}$ . The indices  $J_i$  are spacetime indices and internal indices<sup>5</sup>. The Poincaré and parity invariance requirements impose that the constant tensors  $\lambda_{J_i \dots}$  depend only on  $\eta_{\mu\nu}$  or  $\delta_{\beta}^{\alpha}$ . Thus, the only possible terms are given by the Lorentz-invariant contractions of an undifferentiated antifield  $C^{* \dots}$  with  $\omega^{J_i}$ 's quadratic in the ghosts, contracted with arbitrary internal constant “tensors”. The pure spin-2 and spin-3 terms have already been studied in [4] and [6]. Let us give the exhaustive list of cross-interacting terms :

$$\begin{aligned}
a_2^{(1)} &= f_{A[b c]}^{(1)} C^{* A \mu \nu} C_{\mu}^b C_{\nu}^c d^n x, \\
a_2^{(2)} &= f_{a B c}^{(2)} C^{* a \mu} C_{\mu \nu}^B C^{c \nu} d^n x, \\
a_2^{(3)} &= f_{A B c}^{(3)} C^{* A \mu \nu} \partial_{[\nu} C_{\sigma] \mu}^B C^{c \sigma} d^n x, \\
a_2^{(4)} &= f_{A B c}^{(4)} C^{* A \mu \nu} C_{\mu}^B \partial_{[\nu} C_{\alpha]}^c d^n x, \\
a_2^{(5)} &= f_{a B C}^{(5)} C^{* a \alpha} C^{B \mu \nu} \partial_{[\nu} C_{\alpha] \mu}^C d^n x, \\
a_2^{(6)} &= f_{a b C}^{(6)} C^{* a \mu} \partial^{[\nu} C^{\rho] b} \partial_{[\nu} C_{\rho] \mu}^C d^n x, \\
a_2^{(7)} &= f_{[a b] C}^{(7)} C^{* C \mu \nu} \partial_{[\mu} C_{\alpha]}^a \partial_{[\nu} C_{\beta]}^b \eta^{\alpha \beta} d^n x, \\
a_2^{(8)} &= f_{a B C}^{(8)} C^{* a \mu} \partial^{[\alpha} C^{B | \beta] \nu} \partial_{\mu} \partial_{[\alpha} C_{\beta] \nu}^C d^n x,
\end{aligned}$$

where  $\{f^{(i)}\}_{i=1}^8$  are eight arbitrary constant tensors.

Note that  $a_2^{(8)}$  is trivial when  $n = 3$ , because of a Schouten identity (or equivalently because there is no non-vanishing tensor  $\hat{U}_{\alpha\beta|\mu\nu}$  in dimension 3).

## 4.3 Computation of the gauge transformations

The second set of equations (4.18) has to be satisfied in order for Equation (4.16) to have a solution, and thus in order for  $a_1$  to exist. This is not true for every  $a_2^{(i)}$ . In fact, it is faster to directly compute  $\delta a_2$  and check whether it is  $\gamma$ -exact modulo  $d$ , possibly given a symmetry rule on the internal indices. The latter condition  $\delta a_2 + \gamma a_1 + d c_1 = 0$  implies that the constant tensors  $f_{A b c}^{(1)}$ ,  $f_{a B c}^{(2)}$  and  $f_{a B C}^{(5)}$  must vanish. The following relations between constant tensors are also obtained:  $f_{a B C}^{(8)} = f_{a(B C)}$  and  $f_{A B c}^{(4)} = -\frac{3}{2} f_{A B c}^{(3)}$ . We thus get all of the possible  $a_1$ 's, which we classify

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<sup>5</sup>For example:  $\{\omega^{J_0}\} = \{C^{a\alpha} C^{b\beta}, C^{a\alpha} C^{b\mu\nu}, C^{A\alpha\beta} C^{B\mu\nu}\}$ .

according to the number of derivatives they involve:

$$a_{1,1} = \overset{(3)}{f}_{ABc} \left[ \frac{3}{2} h^{*A\mu\nu\rho} \left( \partial_{[\nu} h_{\sigma]\mu\rho}^B C^{c\sigma} - \partial_{[\nu} C_{\sigma]\mu}^B h_{\rho}^{c\sigma} - h_{\mu\rho}^A{}^\alpha \partial_{[\nu} C_{\alpha]}^c + 3C_{\mu}^{B\alpha} \partial_{[\nu} h_{\alpha]\rho}^c \right) - \frac{3}{4n} h^{*\rho} \partial_{\rho} (h_{\sigma}^B C^{c\sigma}) \right] d^n x, \quad (4.19)$$

$$a_{1,2} = \overset{(6)}{f}_{abC} h^{*a\mu\sigma} \left[ 2\partial^{\nu} h_{\sigma}^{b\rho} \partial_{[\nu} C_{\rho]\mu}^C - \partial^{\nu} C^{b\rho} \partial_{[\nu} h_{\rho]\mu\sigma}^C \right] d^n x + 6 \overset{(7)}{f}_{abC} \left[ h^{*C\mu\nu\rho} - \frac{1}{n} \eta^{\mu\nu} h^{*C\rho} \right] \partial_{[\mu} h_{\alpha]\rho}^a \partial_{[\nu} C_{\beta]}^b \eta^{\alpha\beta} d^n x + \bar{a}_{1,2}, \quad (4.20)$$

$$a_{1,3} = - \overset{(8)}{f}_{aBC} h^{*a\mu\rho} \partial^{\alpha} C^{B\beta\nu} [2\partial_{\mu[\alpha} h_{\beta]\nu\rho}^C - \partial_{\nu[\alpha} h_{\beta]\mu\rho}^C] d^n x + \bar{a}_{1,3}, \quad (4.21)$$

where the  $\bar{a}_{1,i}$  terms are solutions of the homogeneous equations

$$\gamma \bar{a}_{1,i} + d c_{1,i} = 0, \quad (4.22)$$

which is equivalent to solving the equation

$$\gamma \bar{a}_{1,i} = 0 \Rightarrow [\bar{a}_{1,i}] \in H^1(\gamma), \quad (4.23)$$

as proved by Proposition 3.

As a matter of fact, the solutions  $\bar{a}_{1,i}$  linear in the fields play a crucial role in the present analysis, as we show in the next subsection. There is no such solution with one derivative, because the  $\gamma$ -closed functions of the fields involve at least 2 derivatives.

#### 4.4 Computation of cubic vertices

We now have to solve equation (4.17) for each  $a_{1,i}$ . In order to achieve this heavy calculation, we have been using FORM, a powerful software for symbolic computation (see [33]). We have simply considered the most general candidates for  $a_{0,i}$ , implemented the  $a_{1,i}$  (including the most general expression for  $\bar{a}_{1,i}$ ) then solved the systems. It turns out that  $\delta a_{1,1}$  cannot be  $\gamma$ -exact modulo  $d$ , but we obtain consistent vertices for the two other cases, corresponding to  $a_{1,2}$  and  $a_{1,3}$ .

Incidentally, note that the last term on the first line of Equation (4.19) gives the first-order correction needed in order to transform the ordinary derivative  $\partial_{(\mu} \lambda_{\nu\rho)}$  into the covariant one  $\nabla_{(\mu} \lambda_{\nu\rho)}$  for the torsionless metric connection  $\nabla$  associated with  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  (where we omit to write the internal indices). In other words, we recover the result of [16] that both minimal and non-minimal couplings of massless spin-3 field to dynamical gravity in flat space are inconsistent. In fact our result is more general. It says that, whatever the complete gauge transformation of the spin-3 field is, if it contains the first-order correction needed to transform  $\delta_0 h_{\mu\nu\rho} = 3 \partial_{(\mu} \lambda_{\nu\rho)}$  into the covariantized transformation  $\delta_{\lambda} h_{\mu\nu\rho} = 3 \nabla_{(\mu} \lambda_{\nu\rho)}$ , then there is no consistent couplings



between the spin-3 and spin-2 fields in flat background. Apart from Lorentz invariance, we made no assumptions on the form of the possible interaction and imposed no constraints on the number of derivatives (except that it should be finite for locality). The result follows from consistency.

#### 4.4.1 Solution with three derivatives

First, some new relations on the structure constants are obtained:  $f_{[ab]C}^{(7)} = -\frac{1}{3} f_{abC}^{(6)}$ . From now on, we will call the structure constant  $g_{Abc} = g_{A[bc]}$ . The solution of Equations (4.15), (4.16) and (4.17) is as follows :

$$a_{2,2} = g_{Abc} \left[ C^{*b\mu} \partial^\nu C^{c\rho} \partial_{[\nu} C_{\rho]\mu}^A - \frac{1}{3} C^{*A\mu\nu} \partial_{[\mu} C_{\alpha]}^b \partial_{[\nu} C_{\beta]}^c \eta^{\alpha\beta} \right] d^n x, \quad (4.24)$$

$$\begin{aligned} a_{1,2} = & g_{Abc} h^{*b\mu\sigma} \left[ 2\partial^\nu h_{\sigma}^{c\rho} \partial_{[\nu} C_{\rho]\mu}^A - \partial^\nu C^{c\rho} \partial_{[\nu} h_{\rho]\mu\sigma}^A \right] d^n x \\ & - 3g_{Abc} h^{*b\alpha\beta} K_{\alpha\mu\beta\nu}^c C^{A\mu\nu} d^n x \\ & - 2g_{Abc} \left[ h^{*A\mu\nu\rho} - \frac{1}{n} \eta^{\mu\nu} h^{*A\rho} \right] \partial_{[\mu} h_{\alpha]\rho}^b \partial_{[\nu} C_{\beta]}^c \eta^{\alpha\beta} d^n x, \end{aligned} \quad (4.25)$$

The expression for  $a_{0,2}$  is given in Appendix B.

#### 4.4.2 Solution with four derivatives

Let us rename  $f_{aBC}$  the structure constant  $f_{aBC}^{(8)}$ . The solution of the equations (4.15), (4.16) and (4.17) is

$$a_{2,3} = f_{aBC} C^{*a\mu} \partial^\alpha C^{B\beta\nu} \partial_\mu \partial_{[\alpha} C_{\beta]\nu}^C d^n x, \quad (4.26)$$

$$\begin{aligned} a_{1,3} = & f_{aBC} \left[ \frac{3}{8} h^{*a\mu\nu} \partial^\rho F_\rho^B C_{\mu\nu}^C + \frac{3}{2} h^{*B\alpha\beta\gamma} \partial_\alpha K_{\beta\mu\gamma\nu}^a C^{C\mu\nu} + \frac{2}{n} h^{*B\alpha} K_{\alpha\mu\nu\rho}^a \partial^\nu C^{C\rho\mu} \right] d^n x \\ & - f_{aBC} h^{*a\mu\rho} \partial^\alpha C^{B\beta\nu} \left[ 2\partial_{\mu[\alpha} h_{\beta]\nu\rho}^C - \partial_{\nu[\alpha} h_{\beta]\mu\rho}^C \right] d^n x, \end{aligned} \quad (4.27)$$

The expression for  $a_{0,3}$  is given in Appendix B.

## 5 Further results

### 5.1 Conditions at second order in the coupling constants

After this exhaustive determination of the consistent first-order deformations  $W_1$ , it is natural to study the second order equation (2.11). This equation can be decomposed into equations of definite antighost numbers. Let us consider  $W_2 = \int (b_0 + b_1 + b_2 + \dots)$ . The top equation is then

$$(a_2, a_2) = -2\delta b_3 - 2\gamma b_2 + d(\dots),$$

but  $(a_2, a_2)$  does not depend on the fields and on the *antigh*-1 antifields so no  $\delta$ -exact term can appear. This means that  $\forall i > 2 : sb_i = d(\dots)$ . This follows the same pattern as for  $W_1$ . It has been shown in [4] that the homology of  $s$  modulo  $d$  is trivial in that sector and thus the expansion of  $W_2$  stops at *antigh* = 2.

Let us now compute the antibracket  $(a_2, a_2)$ . We have to consider for  $a_2$  the sum of the different terms related to cubic vertices involving spin-2 and/or spin-3 fields. This includes  $a_{2,2}$  and  $a_{2,3}$ , the pure spin-2 Einstein–Hilbert term [4]

$$a_2^{EH} = a_{abc} C^{*a\mu} C^{b\nu} \partial_{[\mu} C_{\nu]}^c d^n x$$

and the two pure spin-3 terms [6]

$$a_2^{BBvD} = k_{BC}^A C_A^{*\mu\nu} (T_{\mu\alpha|\beta}^B T_\nu^{C\alpha|\beta} - 2T_{\mu\alpha|\beta}^B T_\nu^{C\beta|\alpha} + \frac{3}{2} C^{B\alpha\beta} U_{\mu\alpha|\nu\beta}^C) d^n x$$

and

$$a_2^{BBC} = l_{BC}^A C_A^{*\mu\nu} U_{\mu\alpha|\beta\lambda}^B U_\nu^{C\alpha|\beta\lambda} d^n x.$$

Let us give the list of the different antibrackets involving  $a_{2,2}$  and  $a_{2,3}$  in which we have already isolated  $\gamma$ -exact and  $d$ -exact parts :

$$\begin{aligned} (a_2^{EH}, a_{2,2}) &= -a_{cd}^a g_{abE} C^{*c\nu} \partial_{[\nu} C_{\mu]}^d \partial_{[\sigma} C_{\rho]}^b T^{E\sigma\rho|\mu} + a_{cd}^a g_{abE} C_\tau^{*c} C_\mu^d \partial_{[\sigma} C_{\rho]}^b U^{E\tau\mu|\sigma\rho} \\ &\quad + a_{cd}^a g_{ab}^E \partial_{[\tau} C_{\beta]}^c \partial_{[\mu} C_{\nu]}^d \eta^{\nu\beta} \left( C^{*b\alpha} \partial^{[\tau} C_{E\alpha}^{\mu]} + \frac{2}{3} C_E^{*\gamma[\tau} \eta^{\mu]\sigma} \partial_{[\gamma} C_{\sigma]}^b \right) \\ &\quad + \gamma(\dots) + div. \\ (a_2^{EH}, a_{2,3}) &= -a_{ef}^a f_{aBC} C^{*e\nu} \partial_{[\nu} C_{\mu]}^f T_{\rho\sigma|\tau}^B U^{C\rho\sigma|\mu\tau} + a_{ef}^a f_{aBC} C^{*e\tau} C_\mu^f U_{\rho\sigma|\tau\nu}^B U^{C\rho\sigma|\mu\nu} \\ &\quad + \gamma(\dots) + div. \\ (a_{2,2}, a_{2,2}) &= 2g_{aeC} g_{bD}^e \left[ C^{*a\mu} \partial_{[\beta} C_{\gamma]}^b T_{\tau\alpha|\mu}^C U^{D\tau\alpha|\beta\gamma} - \frac{2}{3} C^{*c\mu\tau} \partial_{[\mu} C_{\alpha]}^a \partial_{[\beta} C_{\gamma]}^b U_\tau^{C\alpha|\beta\gamma} \right] \\ &\quad + \gamma(\dots) + div. \\ (a_{2,2}, a_{2,3}) &= g_{abC} f_{DE}^a \left[ C^{*b\nu} T_{\tau\mu|\nu}^C + \frac{2}{3} C_{[\tau}^{*C\nu} \delta_{\mu]}^\sigma \partial_{[\nu} C_{\sigma]}^b \right] U^{D\beta\tau|\rho\alpha} U_{\beta|\rho\alpha}^{E\mu} + \gamma(\dots) + div. \\ (a_{2,2}, a_2^{BBvD}) &= g_{abC} k_{DE}^C C^{*a\mu} \partial_{[\tau} C_{\nu]}^b \times \\ &\quad \left[ \frac{3}{4} U^{D\tau\nu|\alpha\beta} T_{\alpha\beta|\mu}^E - 3U_{\mu(\alpha|\beta)}^D{}^\nu T^{E\tau(\alpha|\beta)} + \frac{3}{4} U_{\mu|\alpha\beta}^D{}^\nu T^{E\alpha\beta|\tau} \right] \\ &\quad + \frac{1}{2} g_{abC} k_{DE}^C C_{\alpha\beta}^{*D} U^{E\alpha\mu|\beta\nu} \partial_{[\mu} C_{\rho]}^a \partial_{[\nu} C_{\sigma]}^b \eta^{\rho\sigma} + \gamma(\dots) + div. \\ (a_{2,3}, a_{2,3}) &= 0 \\ (a_{2,3}, a_2^{BBvB}) &= -f_{aBC} k_{DE}^B C_\tau^{*a} U^{C\rho\mu|\tau\nu} \left[ \frac{9}{8} U_{\rho\mu|\alpha\beta}^D T^{E\alpha\beta}{}_\nu + 3U_{\rho(\alpha|\beta)\nu}^D T_\mu^{D(\alpha|\beta)} \right] \\ &\quad + f_{aBC} k_{DE}^B C^{*a\rho} T^{C\sigma\mu|\nu} \left[ \frac{9}{8} U_{\rho\nu|\alpha\beta}^D U_{\sigma\mu}^{E\alpha\beta} + 3U_{\rho(\alpha|\beta)\sigma}^D U_\mu^{E(\alpha|\beta)}{}_\nu \right] \\ &\quad + \gamma(\dots) + div. \end{aligned}$$

The antibrackets involving  $a_2^{BC}$  are both  $\gamma$ -exact modulo  $d$ , which is mostly because of its high number of derivatives. We can see that  $a_{2,3}$  seems to be behaving well. First, its antibracket with itself is vanishing. Then, looking at its antibracket with  $a_2^{BBvD}$ , it appears that the two expressions between square brackets of non  $\gamma$ -exact terms are very similar. The only difference is the ordering of the internal indices and thus, by imposing the relation  $f_{aB(C}k_{D)E}^B = 0$ , this antibracket is consistent. Then, we can see that  $a_{2,2}$  seems to be obstructed. The antibrackets  $(a_{2,2}, a_{2,2})$  and  $(a_{2,2}, a_2^{BBvD})$  have some non  $\gamma$ -exact terms in common but there is no combination allowing to remove them all. The only remaining ways to remove such terms would be to consider  $a_2$  terms involving combinations of spins  $1-2-2$ ,  $1-3-3$ ,  $1-2-3$  or  $2-3-4$ . As we stated in the introduction, it is already known that there is no deformation of the algebra for combinations  $s_1 - s_2 - s_2$  where  $s_1 \geq 2s_2$  so there is no  $a_2$  for a  $4-2-2$  combination. It is very easy to see that there is no Lorentz-invariant way to write an  $a_2$  for a  $1-2-3$  combination. The remaining cases are more promising and we intend to study them in a near future. But it already appears to us that they do not remove the obstruction in  $(a_{2,2}, a_{2,2})$ . It is not very surprising that the antibrackets  $(a_2^{EH}, a_{2,2})$  and  $(a_{2,2}, a_{2,3})$  do not behave well either. The last term,  $(a_2^{EH}, a_{2,3})$ , brings in another interesting feature. Once again, the obstructions that it brings could be eliminated, resorting to  $2-3-4$  interactions. However, as a preliminary computation shows, no terms quadratic in  $U$  appear for  $2-3-4$  deformations. So this indicates that the  $2-3-3$  deformation that we have found is not compatible with the Einstein–Hilbert deformation.

## 5.2 Gauging of rigid symmetries

In the previous section, we obtained two classes of first-order deformations associated with non-Abelian algebras. Corresponding to the gauge-algebra deformation  $a_{2,2}$  displayed in equation (4.24) we have the following first-order gauge transformations

$$\delta h_{\alpha\beta}^a = g_B^a{}_{\bar{b}} \left( 3K_{\mu\alpha\beta\nu}^b \lambda^{B\mu\nu} + \partial_{[\nu} h_{\rho]\alpha}^b \partial^\nu \lambda_{\beta}^{B\rho} + \partial_{[\nu} h_{\rho]\beta}^b \partial^\nu \lambda_{\alpha}^{B\rho} - \partial_{[\nu} h_{\rho]\alpha\beta}^B \partial^\nu \epsilon^{b\rho} \right), \quad (5.28)$$

$$\delta h_{\mu\nu\rho}^A = -2g_{bc}^A \left( \partial_{[\mu} h_{\sigma]\rho}^b \partial_{[\nu} \epsilon_{\tau]}^c \eta^{\sigma\tau} - \frac{1}{n} \eta_{\mu\nu} \partial_{[\alpha} h_{\beta]\rho}^b \partial^{[\alpha} \epsilon^{\beta]c} \right), \quad (5.29)$$

where the right-hand-side of (5.29) must be totally symmetrized on the free indices. The first term on the right-hand-side of (5.28) comes from  $\bar{a}_{1,2}$ , solution of the equation (4.23). The term  $\bar{\delta}_\lambda h_{\alpha\beta}^a = -3g_{bB}^a K_{\mu\alpha\nu\beta}^b \lambda^{B\mu\nu}$  is therefore absolutely necessary for the existence of the cubic vertex associated with  $a_{2,2}$ . Up to some trivial gauge transformation, it is possible to express  $\bar{\delta}_\lambda h_{\alpha\beta}^a$  differently:  $\bar{\delta}_\lambda h_{\alpha\beta}^a = \frac{3}{2} \mathcal{S}_\lambda h_{\alpha\beta}^a - 3\partial_{(\alpha} V_{\beta)}^a$ , where

$$\mathcal{S}_\lambda h_{\alpha\beta}^a = g_{bB}^a \left[ \lambda^{B\mu\nu} \partial_\mu \partial_\nu h_{\alpha\beta}^b + \partial_\alpha \lambda^{B\mu\nu} \partial_\mu h_{\nu\beta}^b + \partial_\beta \lambda^{B\mu\nu} \partial_\mu h_{\nu\alpha}^b + \partial_\alpha \partial_\beta \lambda^{B\mu\nu} h_{\mu\nu}^b \right] \quad (5.30)$$

and  $V_\alpha^a = g_{bB}^a [2\lambda^{B\mu\nu} \partial_{[\mu} h_{\alpha]\nu}^b + \frac{1}{2} \partial_\alpha (\lambda^{B\mu\nu} h_{\mu\nu}^b)]$ . The transformations  $\frac{3}{2} \mathcal{S}_\lambda h_{\alpha\beta}^a$  and  $\bar{\delta}_\lambda h_{\alpha\beta}^a$  are identified since they differ by a trivial zeroth-order gauge transformation  $-3\partial_{(\alpha} V_{\beta)}^a$  which can be eliminated by a redefinition of the gauge parameters  $\epsilon_\alpha^a$  in Formula (2.2). Because the transformation  $\bar{\delta}_\lambda h_{\alpha\beta}^a$  involves the spin-2 fields  $h_{\alpha\beta}^a$  only *via* the linearized Riemann tensors  $K_{\mu\alpha\nu\beta}^b$ , it is clear that  $\bar{\delta}_\lambda h_{\alpha\beta}^a$  is  $\gamma$ -closed and the following commutation relations hold  $[\delta_\epsilon, \bar{\delta}_\lambda] = 0$ , where  $\delta_\epsilon h_{\alpha\beta}^a = 2\partial_{(\alpha} \epsilon_{\beta)}^a$ . The latter vanishing of commutator simply re-expresses the fact that  $\bar{a}_1$  is not derived from any algebra-deformation  $a_2$ , since it satisfies the equation (4.22)  $\gamma\bar{a}_1 + \delta a_2 = dc_1$  with  $a_2$   $\delta$ -trivial modulo  $d$ .

When the parameters  $\lambda^{A\mu\nu}$  are all constant, the transformations (5.28) reduce, modulo trivial gauge transformations, to  $\delta h_{\mu\nu}^a = \frac{3}{2} g_{bB}^a \lambda^{B\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu}^b$ . Therefore, we recover the rigid symmetries  $\delta h_{\mu_1 \dots \mu_s}^a \rightarrow h_{\mu_1 \dots \mu_s}^a + \xi_c^a \alpha_1 \dots \alpha_n \partial_{\alpha_1} \dots \partial_{\alpha_n} h_{\mu_1 \dots \mu_s}^c$  exhibited in [15], where  $s = 2$ ,  $n = 2$  and  $\xi_{\alpha_1 \alpha_2}^{ac} = \frac{3}{2} \lambda_{\alpha_1 \alpha_2}^B \delta^{cb} g_B^a$ . As in [15], we have  $\xi_{\alpha_1 \alpha_2}^{ac} = (-)^{n-1} \xi_{\alpha_1 \alpha_2}^{ca}$  due to the symmetry properties of  $g_{Abc}$  that we derived in our cohomological analysis in Section 4.4.1. Retrospectively, we can therefore consider the consistent first-order deformation obtained in Section 4.4.1 as resulting from the gauging of the rigid symmetry  $h_{\mu_1 \mu_2}^a \rightarrow h_{\mu_1 \mu_2}^a + \xi_c^a \alpha_1 \alpha_2 \partial_{\alpha_1} \partial_{\alpha_2} h_{\mu_1 \mu_2}^c$  presented in [15]. However, we have shown that this gauging is inconsistent when pushed up to the second order in the coupling constants.

**Remark:** In the framework of differential multicomplexes (see [26] and the appendix A of [27] for precise definitions and related concepts) we are given  $s$  differential forms  $d_i x^\mu$  ( $i = 1, 2, \dots, s$ ) obeying  $d_i x^\mu d_j x^\nu = (-)^{\delta_{ij}} d_j x^\nu d_i x^\mu$ , where wedge and symmetric products are not explicitly written. Their adjoints  $(d_i x^\mu)^\dagger$  obey the same (anti)commutation relations and we have the crossed (anti)commutation relations  $[d_i x^\mu, (d_j x^\nu)^\dagger]_\pm = \delta_{ij} \eta^{\mu\nu}$  where  $[\ , \ ]_\pm$  stands for the graded commutator. The  $s$  nilpotent differential operators  $d_i = d_i x^\mu \frac{\partial^L}{\partial x^\mu}$  ( $i = 1, 2, \dots, s$ ) are defined which generalize the exterior differential of the de Rham complex. Setting  $s = 2$  and forgetting about the internal indices for a while since they play no role in the following discussion, we may view  $h_{\alpha\beta}$  as the components of the differential multiform  $h = h_{\alpha\beta} d_1 x^\alpha d_2 x^\beta$  and re-write Formula (5.30) as

$$\begin{aligned} \mathcal{S}_\lambda h &= (i_\lambda d_1 d_2 + d_1 i_\lambda d_2 + d_2 i_\lambda d_1 + d_1 d_2 i_\lambda) h \\ &= (d_1 + d_2 + i_\lambda)^3 h \end{aligned} \quad (5.31)$$

where  $i_\lambda = \lambda_{\mu\nu} (d_1 x^\mu)^\dagger (d_2 x^\nu)^\dagger$ . The expression (5.31)  $\mathcal{S}_\lambda = (d_1 + d_2 + i_\lambda)^3$  generalizes Cartan's formula  $\mathcal{L}_\zeta = (i_\zeta + d)(i_\zeta + d) = i_\zeta d + d i_\zeta$  for the usual Lie derivative along the vector  $\zeta$  in the framework of the de Rham differential complex. There,  $i_\zeta$  is the interior product associated with the vector  $\zeta$  and  $d$  the usual exterior differential.

When a spin-1 field  $A_\mu$  is coupled to gravity, it transforms under diffeomorphisms via the Lie derivative along the diffeomorphism vector  $\zeta$ ,  $\delta_\zeta^{diff} A = \mathcal{L}_\zeta A$  where  $A = dx^\mu A_\mu$ . By analogy, it is

tempting to view  $\mathcal{S}_\lambda h$  as the “spin-3 diffeomorphism” of the spin-2 field  $h = h_{\alpha\beta} d_1 x^\alpha d_2 x^\beta$  along the gauge parameter  $\lambda = \lambda^{\mu\nu} \partial_\mu \partial_\nu$  of the spin-3 field. However, as we said before, this transformation appears in the deformation  $W_1 = \int (a_{2,2} + a_{1,2} + a_{0,2})$  which, we have shown, is inconsistent when pushed at second order in the coupling constants.

Let us now consider the gauge transformations corresponding to the four-derivative deformation  $W_1 = \int (a_{2,3} + a_{1,3} + a_{0,3})$  obtained in Section 4.4.2. See in particular Equation (4.27). It turns out to be convenient to add trivial gauge transformations to them, by noting that

$$\begin{aligned} \frac{3}{8} h^{*a\mu\nu} \partial^\rho F_\rho^B C_{\mu\nu}^C &= -\frac{3}{4n} h^{*a\mu\nu} \delta(\partial^\rho h_\rho^{*B}) C_{\mu\nu}^C = \delta\left(\frac{3}{4n} h^{*a\mu\nu} \partial^\rho h_\rho^{*B} C_{\mu\nu}^C\right) + \dots \\ &= -\frac{3}{2n} h^{*B\rho} \left(\partial_\rho K_{\mu\nu}^a C^{C\mu\nu} + \frac{4}{3} K^{a\mu\nu} \partial_{[\rho} C_{\mu]\nu}^C\right) + \delta(\dots) + \gamma(\dots) + d(\dots). \end{aligned}$$

Accordingly, we have the following first-order gauge transformations of the spin-2 and spin-3 fields:

$$\delta h_{\mu\nu}^a = -f_{BC}^a \left[ \partial^\alpha \lambda^{B\beta\sigma} \left( \partial_{\mu[\alpha} h_{\beta]\sigma\nu}^C + \partial_{\nu[\alpha} h_{\beta]\sigma\mu}^C - \partial_{\sigma[\alpha} h_{\beta]\mu\nu}^C \right) \right] \quad (5.32)$$

$$\begin{aligned} \delta h_{\alpha\beta\gamma}^B &= f_a^B C^a \left[ \frac{3}{2} \left( \partial_\alpha K_{\beta\mu\gamma\nu}^a - \frac{1}{n} \eta_{\alpha\beta} \partial_\gamma K_{\mu\nu}^a \right) \lambda^{C\mu\nu} \right. \\ &\quad \left. + \frac{2}{n} \eta_{\alpha\beta} \left( K_{\gamma\mu\nu\rho}^a + K_{\mu[\nu} \eta_{\rho]\gamma}^a \right) \partial^\nu \lambda^{C\rho\mu} \right] \quad (5.33) \end{aligned}$$

where the right-hand-side of the second equation must be totally symmetrized over the free indices. By setting the gauge parameters to constants, one could wonder whether some rigid symmetry appears, that could retrospectively be seen as being gauged. Clearly, nothing comes from the gauge transformations of  $h_{\alpha\beta}^a$ . As far as the spin-3 fields are concerned, only the term  $f_a^B C^a \left[ \frac{3}{2} \left( \partial_\alpha K_{\beta\mu\gamma\nu}^a - \frac{1}{n} \eta_{\alpha\beta} \partial_\gamma K_{\mu\nu}^a \right) \lambda^{C\mu\nu} \right]$  (which must be symmetrized over  $\alpha\beta\gamma$ ) survives when the gauge parameters are set to constants. However, with  $\lambda^{C\mu\nu}$  all constants, it is readily seen that it can be written as  $3 \partial_{(\alpha} \Lambda_{\beta\gamma)}^B$  where  $\Lambda_{\beta\gamma}^B = -f_a^B C^a \left[ \frac{1}{2} \left( K_{\mu(\beta\gamma)\nu}^a + \frac{1}{n} \eta_{\beta\gamma} K_{\mu\nu}^a \right) \lambda^{C\mu\nu} \right]$  and hence can be eliminated by a redefinition of the gauge parameters of the free gauge transformations (2.5). Indeed,  $\eta^{\beta\gamma} \Lambda_{\beta\gamma}^B$  is vanishing. As a result, no rigid gauge transformations can be obtained upon setting to constants the gauge parameters in the first-order gauge transformations (5.32) and (5.33). [Note, however, that the constants are not the only solutions of the higher-spin Killing equations.]

## 6 Conclusions and perspectives

In this paper we carefully analyzed the problem of introducing consistent cross-interactions among a countable collection of spin-3 and spin-2 gauge fields in flat spacetime of arbitrary dimension  $n > 3$ . For this purpose we used the powerful BRST cohomological deformation techniques in order to be exhaustive. Under the sole assumptions of locality, parity invariance, Poincaré invariance and perturbative deformation of the free theory, we proved that only two classes of non-Abelian

deformations are consistent at first order. The first deformation, which involves three derivatives in the Lagrangian, was already mentioned in the work [14]. We showed that it is obstructed at second order in the deformation parameters if no other fields are present in the analysis. The second deformation involves four derivatives in the Lagrangian and passes the second-order constraint — equivalent to the Jacobi identities of the gauge algebra at the corresponding order — where the previous deformation failed. Moreover, combining this algebra-deformation with the one corresponding to the Berends–Burgers–van Dam deformation, the crossed second-order constraint is also satisfied given a symmetry condition on the product of the internal coefficients, while a combination with the Einstein–Hilbert deformation is obstructed.

We also discussed the link between the gauge transformations associated with the three-derivative vertex and the rigid symmetries of the free theory exhibited in [15]. More precisely, these gauge transformations can be seen as a gauging of these rigid symmetries, similarly to what happens in the “Bell–Robinson” cases  $s_1 - s_2 - s_2$  where  $s_1 \geq 2s_2$  [15, 17].

It would be of interest to enlarge the set of fields to spin 4 and see if this allows to remove the previous obstructions at order two. A hint that this might be sufficient comes from the fact that the commutator of two spin-3 generators produces spin-2 and spin-4 generators for the bosonic higher-spin algebra of Ref. [2]. We hope to address this issue in the future.

More generally, we believe that the two consistent vertices exhibited here can be related to the flat space limit (appropriately defined, in order to avoid potential problems related to the non-analyticity in the cosmological constant  $\Lambda$ ) of the spin-3–spin-2 sector of the full  $AdS_n$  higher-spin gauge theory of [2, 21, 22] (and references therein). Such a connection would provide a geometric meaning for the long expressions for the vertices. That such a Minkowski–(A)dS link should be possible was mentioned in the pure spin-3 case [6]. In fact, it can be shown that there is a correspondence between the non-Abelian gauge algebras obtained in the current flat-spacetime setting and the  $AdS_n$  higher-spin algebra  $hu(1|2 : [n - 1, 2])$  reviewed e.g. in the second reference of [21]. This is beyond the scope of the present work and will be reported elsewhere.

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## A Proof of Proposition 1

First, let us recall the results for pure spin-2 [4] and pure spin-3 [6] theories. In the spin-2 case, a convenient set of representatives of the cohomology of  $\gamma$  is the set of functions of the antighosts,  $[K^a]$ ,  $C_\mu^a$  and  $\partial_{[\mu} C_{\nu]}^a$ . In the spin-3 case, the natural set of representatives of the cohomology of  $\gamma$  is the set of functions of the antighosts,  $[K^A]$ ,  $[F]$ ,  $C_{\mu\nu}^A$ ,  $\hat{T}^A$  and  $\hat{U}^A$ . Proposition 1 just says that, in the mixed case, the cohomology of  $\gamma$  is the direct product of the previous two sets.

*pgh* 0

$\gamma$  can be seen as the sum of its spin 2 and spin 3 restrictions, that we will note  $\gamma_2$  and  $\gamma_3$ . The homology of  $\gamma_2$  (resp.  $\gamma_3$ ) is simply the direct product of the homology in the spin 2 (resp. 3) case and the set of all spin 3 (resp. 2) fields. Given an arbitrary  $\gamma$ -closed function  $f$  at *pgh* 0 (i.e. it does not depend on any ghost), we have  $\gamma f = \gamma_2 f + \gamma_3 f = 0$ .

But, as  $\gamma_2 f$  is linear in the spin 2 ghosts and  $\gamma_3 f$  is linear in the spin 3 ghosts, the two terms are linearly independant and thus both vanish. This means that  $f$  is in the intersection of the homologies of  $\gamma_2$  and  $\gamma_3$ , so  $f = f([\Phi_I^*], [K^a], [K^A], [F^A])$ . As there are no  $\gamma$ -exact objects in *pgh* 0,  $H^0(\gamma)$  is the set of those functions.

*pgh* > 0

Let us denote generically  $\mathcal{C}^a/\mathcal{C}^A$  a basis of the spin 2/spin 3 ghosts and their derivatives. The  $\gamma$ -exact ghosts will be noted  $\bar{\mathcal{C}}^a = \gamma[h]^a$  and  $\bar{\mathcal{C}}^A = \gamma[h]^A$  (the bracketed fields are the adequate combinations of the fields or some of their derivatives). A *pgh*  $i$  object  $f^i$  (with *antigh*  $f^i = k$ ) is then a linear combination:

$$f^i = \sum_j f_{a_1 \dots a_j A_1 \dots A_{i-j}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}}$$

Imposing that  $\gamma f^i = 0$  gives rise to some relations between the  $\gamma f_{a_1 \dots a_j A_1 \dots A_{i-j}}$ . In general, they would be a combination of the  $\bar{\mathcal{C}}^a$  and the  $\bar{\mathcal{C}}^A$ , but the coefficients will have to take particular values:

$$\begin{aligned} \exists \{ \mathcal{K} \} \mid \gamma f_{a_1 \dots a_j A_1 \dots A_{i-j}} &= (-1)^k \mathcal{K}_{a_1 \dots a_{j+1} A_1 \dots A_{i-j}}^{(j+1)} \bar{\mathcal{C}}^{a_{j+1}} \\ &+ (-1)^{i+j+k} \mathcal{K}_{a_1 \dots a_j A_1 \dots A_{i-j+1}}^{(j)} \bar{\mathcal{C}}^{A_{i-j+1}} \end{aligned}$$

The coefficients  $\mathcal{K}$  must be taken such that at least one index  $a$  and one index  $A$  are contracted only with gamma exact objects (say the last index of both kinds as in the last equation). The *antigh* sign factor has been introduced for later convenience. Finally, the  $j = 0$  and  $j = i + 1$

coefficients have to vanish.

Let us remark that  $\gamma^2 f_{a_1 \dots a_j A_1 \dots A_{i-j}} = 0$  implies that  $\forall j : \gamma \mathcal{K}^{(j)} = 0$ . This means that

$$\begin{aligned} f_{a_1 \dots a_j A_1 \dots A_{i-j}} &= \mathcal{K}_{a_1 \dots a_{j+1} A_1 \dots A_{i-j}}^{(j+1)} [h]^{a_{j+1}} + (-1)^{i+j} \mathcal{K}_{a_1 \dots a_j A_1 \dots A_{i-j+1}}^{(j)} [h]^{A_{i-j+1}} \\ &\quad + g_{a_1 \dots a_j A_1 \dots A_{i-j}} \end{aligned}$$

where  $g_{a_1 \dots a_j A_1 \dots A_{i-j}} \in H^0(\gamma)$  and we obtain an expression for  $f^i$  itself:

$$\begin{aligned} f^i &= \sum_j \left[ \mathcal{K}_{a_1 \dots a_{j+1} A_1 \dots A_{i-j}}^{(j+1)} [h]^{a_{j+1}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} \right. \\ &\quad \left. + (-1)^{i+j} \mathcal{K}_{a_1 \dots a_j A_1 \dots A_{i-j+1}}^{(j)} [h]^{A_{i-j+1}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} \right. \\ &\quad \left. + g_{a_1 \dots a_j A_1 \dots A_{i-j}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} \right] \\ &= \sum_j \left[ \mathcal{K}_{a_1 \dots a_{j+1} A_1 \dots A_{i-j}}^{(j+1)} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j-1}} ([h]^{a_{j+1}} \mathcal{C}^{A_{i-j}} + \mathcal{C}^{a_{j+1}} [h]^{A_{i-j}}) \right. \\ &\quad \left. + g_{a_1 \dots a_j A_1 \dots A_{i-j}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} \right] \\ &= \gamma \left( (-1)^{i+k+1} \sum_j \mathcal{K}_{a_1 \dots a_{j+1} A_1 \dots A_{i-j}}^{(j)} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j-1}} [h]^{a_{j+1}} [h]^{A_{i-j}} \right) \\ &\quad + \sum_j g_{a_1 \dots a_j A_1 \dots A_{i-j}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} \end{aligned}$$

The first term of the last expression is trivial in  $H^i(\gamma)$  while the second, as we announced, depends on the fields only through  $[K^a]$ ,  $[K^A]$  and  $[F^A]$  thanks to the fact that the coefficients  $g$  belong to  $H^0(\gamma)$ .

Finally, the second term can be rewritten as

$$\begin{aligned} g_{a_1 \dots a_j A_1 \dots A_{i-j}} \mathcal{C}^{a_1} \dots \mathcal{C}^{a_j} \mathcal{C}^{A_1} \dots \mathcal{C}^{A_{i-j}} &= G_a \mathcal{C}^a + G_A \mathcal{C}^A + \alpha_I \omega^I \\ &= \gamma \{ G_a [h]^a + G_A [h]^A \} + \alpha_I \omega^I \end{aligned}$$

where, as stated before,  $\{\omega^J\}$  is a basis of the products of non-exact ghosts. The only non-exact term in the last equation is the last one, with  $\alpha_J \in H^0(\gamma)$ . This expression is the general form for a representative of  $H^i(\gamma)$  that we announced.

## B First-order vertices

The three-derivative first-order vertex corresponding to the algebra deformation  $a_{2,2}$  given in (4.24) is

$$a_{0,2} = \mathcal{L}^{(3)} d^n x = g_{bc}^A U_A^{bc} d^n x,$$



where, denoting  $h = \eta^{\mu\nu} h_{\mu\nu}$  and  $h_\alpha = \eta^{\mu\nu} h_{\alpha\mu\nu}$ ,

$$\begin{aligned}
U_A^{bc} = & -\frac{1}{2} h_A^\alpha \square h^b \partial_\alpha h^c + \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\beta \partial_\gamma h^b \partial_\alpha h^c + \frac{1}{2} h_A^{\alpha\beta\gamma} \square h_{\beta\gamma}^b \partial_\alpha h^c \\
& + \frac{1}{2} h_A^\alpha \partial^\beta \partial^\gamma h_{\beta\gamma}^b \partial_\alpha h^c + h_A^\alpha \partial^\beta \partial^\gamma h^b \partial_\alpha h_{\beta\gamma}^c + \frac{1}{2} h_A^\alpha \square h^{b\beta\gamma} \partial_\alpha h_{\beta\gamma}^c \\
& - h_A^{\alpha\beta\gamma} \partial_{\beta\delta} h_\gamma^{b\delta} \partial_\alpha h^c - h_A^{\alpha\beta\gamma} \partial_{\beta\delta} h^b \partial_\alpha h_\gamma^{c\delta} - \frac{1}{2} h_A^{\alpha\beta\gamma} \square h_{\beta\delta}^b \partial_\alpha h_\gamma^{c\delta} \\
& - \frac{3}{2} h_A^\alpha \partial^\beta \partial^\gamma h_{\beta\delta}^b \partial_\alpha h_\gamma^{c\delta} - \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\beta \partial_\gamma h^{b\mu\nu} \partial_\alpha h_{\mu\nu}^c - h_A^{\alpha\beta\gamma} \partial^\mu \partial^\nu h_{\beta\gamma}^b \partial_\alpha h_{\mu\nu}^c \\
& + \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\gamma \partial_\delta h^{b\delta\varepsilon} \partial_\alpha h_{\beta\varepsilon}^c + \frac{3}{2} h_A^{\alpha\beta\gamma} \partial_\gamma \partial_\delta h_{\beta\varepsilon}^b \partial_\alpha h^{c\delta\varepsilon} + h_A^{\alpha\beta\gamma} \partial^\delta \partial^\varepsilon h_{\beta\delta}^b \partial_\alpha h_{\gamma\varepsilon}^c \\
& - \frac{1}{4} h_A^\alpha \partial_\alpha \partial_\gamma h^b \partial^\gamma h^c - \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\alpha \partial_\varepsilon h_{\beta\gamma}^b \partial^\varepsilon h^c + h_A^{\alpha\beta\gamma} \partial_\alpha \partial_\varepsilon h^b \partial^\varepsilon h_{\beta\gamma}^c \\
& + \frac{1}{4} h_A^\alpha \partial_\alpha \partial_\varepsilon h^{b\mu\nu} \partial^\varepsilon h_{\mu\nu}^c - \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\alpha \partial_\varepsilon h_{\beta\delta}^b \partial^\varepsilon h_\gamma^{c\delta} + h_{A\mu} \partial_\alpha \partial_\varepsilon h^{b\alpha\mu} \partial^\varepsilon h^c \\
& - h_{A\mu\beta\gamma} \partial_\alpha \partial_\varepsilon h^{b\alpha\mu} \partial^\varepsilon h^{c\beta\gamma} + \frac{1}{2} h_A^\tau \partial_\alpha \partial_\varepsilon h^{b\alpha\mu} \partial^\varepsilon h_{\mu\tau}^c - h_{A\mu} \partial_\alpha \partial_\varepsilon h^b \partial^\varepsilon h^{c\alpha\mu} \\
& + h_{A\mu\beta\gamma} \partial_\alpha \partial_\varepsilon h^{b\beta\gamma} \partial^\varepsilon h^{c\alpha\mu} - \frac{1}{2} h_A^\tau \partial_\alpha \partial_\varepsilon h_{\mu\tau}^b \partial^\varepsilon h^{c\alpha\mu} - \frac{1}{2} h_A^\alpha \square h_{\alpha\gamma}^b \partial^\gamma h^c \\
& + \frac{1}{2} h_A^{\alpha\beta\gamma} \partial_\beta \partial_\gamma h_{\alpha\rho}^b \partial^\rho h^c + \frac{1}{2} h_A^{\alpha\beta\gamma} \square h_{\alpha\rho}^b \partial^\rho h_{\beta\gamma}^c + \frac{1}{2} h_A^\alpha \partial_\beta \partial_\gamma h_{\alpha\rho}^b \partial^\rho h^{c\beta\gamma} \\
& - h_A^{\alpha\beta\gamma} \partial_\beta \partial_\delta h_{\alpha\rho}^b \partial^\rho h_\gamma^{c\delta} - \frac{1}{4} h_{A\mu} \partial^\beta \partial^\mu h_{\beta\gamma}^b \partial^\gamma h^c - \frac{1}{2} h_{A\mu\nu\rho} \partial^\beta \partial^\mu h_{\beta\gamma}^b \partial^\gamma h^{c\nu\rho} \\
& + h_{A\mu}^\nu \partial^\beta \partial^\mu h_{\beta\gamma}^b \partial^\gamma h_{\mu\nu}^c - \frac{1}{2} h_{A\mu} \square h_{\beta\gamma}^b \partial^\gamma h^{c\beta\mu} + \frac{1}{2} h_{A\mu\nu\rho} \partial^\nu \partial^\rho h_{\beta\gamma}^b \partial^\gamma h^{c\beta\mu} \\
& - \frac{1}{4} h_A^\tau \partial_\mu \partial_\tau h_{\beta\gamma}^b \partial^\gamma h^{c\beta\mu} + \frac{1}{2} h_A^\alpha \square h^b \partial^\gamma h_{\alpha\gamma}^c - \frac{1}{2} h_A^{\alpha\mu\nu} \partial_\mu \partial_\nu h^b \partial^\gamma h_{\alpha\gamma}^c \\
& - \frac{1}{2} h_A^{\alpha\mu\nu} \square h_{\mu\nu}^b \partial^\gamma h_{\alpha\gamma}^c - \frac{1}{2} h_A^\alpha \partial^\mu \partial^\nu h_{\mu\nu}^b \partial^\gamma h_{\alpha\gamma}^c + h_A^{\alpha\mu\nu} \partial_\nu \partial_\rho h_\mu^{b\rho} \partial^\gamma h_{\alpha\gamma}^c \\
& + \frac{1}{4} h_{A\beta} \partial^\alpha \partial^\beta h^b \partial^\gamma h_{\alpha\gamma}^c + \frac{1}{2} h_{A\beta\mu\nu} \partial^\alpha \partial^\beta h^{b\mu\nu} \partial^\gamma h_{\alpha\gamma}^c - h_A^\tau \partial^\alpha \partial^\mu h_{\mu\tau}^b \partial^\gamma h_{\alpha\gamma}^c \\
& + \frac{1}{2} h_{A\mu} \square h^{b\alpha\mu} \partial^\gamma h_{\alpha\gamma}^c - \frac{1}{2} h_{A\mu\nu\rho} \partial^\nu \partial^\rho h^{b\alpha\mu} \partial^\gamma h_{\alpha\gamma}^c + \frac{1}{4} h_A^\tau \partial_\mu \partial_\tau h^{b\alpha\mu} \partial^\gamma h_{\alpha\gamma}^c.
\end{aligned}$$

The four-derivative first-order vertex corresponding to the algebra deformation  $a_{2,3}$  given in (4.26) is

$$a_{0,3} = \mathcal{L}^{(3)} d^n x = f_{BC}^a T_a^{BC} d^n x,$$

where

$$\begin{aligned}
T_a^{BC} = & \frac{1}{4} h_a \square \partial_{\alpha\beta}^2 h_\gamma^B h^{C\alpha\beta\gamma} - \frac{1}{4} h_a \partial_{\alpha\beta\gamma\delta}^4 h^{B\alpha\beta\varepsilon} h_\varepsilon^{C\gamma\delta} + \frac{1}{8} h_a \partial_{\alpha\beta\gamma\delta}^4 h^{B\alpha} h^{C\beta\gamma\delta} \\
& - \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\beta\mu\nu}^4 h_\gamma^B h^{C\gamma\mu\nu} - \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\mu\nu\rho}^4 h_\beta^B h^{C\mu\nu\rho} + \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\mu\nu\rho}^4 h_\beta^{B\rho\sigma} h_\sigma^{C\mu\nu} \\
& + \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu\nu\rho}^4 h^{B\mu} h_\beta^{C\nu\rho} - \frac{1}{4} h_a^{\alpha\beta} \square \partial_{\mu\nu}^2 h_{\alpha\beta\rho}^B h^{C\mu\nu\rho} + \frac{1}{4} h_a^{\alpha\beta} \partial_{\mu\nu\rho\sigma}^4 h^{B\mu} h_{\alpha\beta}^{C\nu\rho\sigma} \\
& + \frac{1}{2} h_a^{\alpha\beta} \square \partial_{\mu\nu}^2 h_\alpha^B h_\beta^{C\mu\nu} - \frac{1}{2} h_a^{\alpha\beta} \partial^{4\mu\nu\rho\sigma} h_{\alpha\mu\nu}^B h_{\beta\rho\sigma}^C - \frac{3}{8} h_a^{\alpha\beta} \square \partial^{2\mu\nu} h_\mu^B h_{\alpha\beta\nu}^C
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} h_a^{\alpha\beta} \partial^{4\mu\nu\rho\sigma} h_{\mu\nu\rho}^B h_{\alpha\beta\sigma}^C \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial_{\beta\mu\nu}^3 h_\rho^B \partial_\alpha h^{C\mu\nu\rho} + \frac{3}{8} h_a^{\alpha\beta} \partial_{\beta\mu\nu}^3 h^{B\mu} \partial_\alpha h^{C\nu} + \frac{1}{4} h_a^{\alpha\beta} \partial_{\beta\mu\nu}^3 h^{B\mu\rho\sigma} \partial_\alpha h_{\rho\sigma}^{C\nu} \\
& + \frac{3}{4} h_a^{\alpha\beta} \square \partial_\gamma h_\beta^B \partial_\alpha h^{C\gamma} - \frac{1}{4} h_a^{\alpha\beta} \square \partial_\gamma h_{\beta\mu\nu}^B \partial_\alpha h^{C\gamma\mu\nu} - \frac{1}{4} h_a^{\alpha\beta} \partial_{\mu\nu\rho}^3 h_\beta^B \partial_\alpha h^{C\mu\nu\rho} \\
& - \frac{3}{4} h_a^{\alpha\beta} \partial^{3\mu\nu\rho} h_{\beta\nu\rho}^B \partial_\alpha h_\mu^C + \frac{1}{2} h_a^{\alpha\beta} \partial^{3\mu\nu\rho} h_{\beta\mu\sigma}^B \partial_\alpha h_{\nu\rho}^{C\sigma} - \frac{9}{8} h_a^{\alpha\beta} \square \partial_\gamma h^{B\gamma} \partial_\alpha h_\beta^C \\
& + \frac{3}{4} h_a^{\alpha\beta} \partial_{\mu\nu\rho}^3 h^{B\mu\nu\rho} \partial_\alpha h_\beta^C + \frac{1}{2} h_a \square \partial^\alpha h^{B\mu\nu\rho} \partial_\alpha h_{\mu\nu\rho}^C + \frac{1}{2} h_a \partial^{3\alpha\mu\nu} h^{B\rho} \partial_\alpha h_{\mu\nu\rho}^C \\
& + \frac{3}{8} h_a \partial^{3\alpha\mu\nu} h_\mu^B \partial_\alpha h_\nu^C - h_a \partial^{3\alpha\mu\nu} h_{\mu\rho\sigma}^B \partial_\alpha h_\nu^{C\rho\sigma} - \frac{1}{2} h_a^{\alpha\beta} \partial_{\gamma\alpha\beta}^3 h_{\rho\sigma\tau}^B \partial^\gamma h^{C\rho\sigma\tau} \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial_{\gamma\mu\nu}^3 h_{\alpha\beta\rho}^B \partial^\gamma h^{C\rho\sigma\tau} + \frac{3}{4} h_a^{\alpha\beta} \partial^{3\gamma\mu\nu} h_{\alpha\beta\mu}^B \partial_\gamma h_\nu^C - \frac{1}{2} h_a^{\alpha\beta} \partial^{3\gamma\mu\nu} h_\mu^B \partial_\gamma h_{\alpha\beta\nu}^C \\
& - \frac{3}{2} h_a^{\alpha\beta} \partial_{\gamma\alpha\mu}^3 h_\beta^B \partial^\gamma h^{C\mu} + h_a^{\alpha\beta} \partial_{\gamma\alpha\mu}^3 h_{\beta\nu\rho}^B \partial^\gamma h^{C\mu\nu\rho} + \frac{3}{8} h_a^{\alpha\beta} \partial_{\gamma\alpha\mu}^3 h^{B\mu} \partial^\gamma h_\beta^C \\
& + \frac{3}{4} h_a^{\alpha\beta} \partial_{\gamma\alpha\mu}^3 h^{B\mu\nu\rho} \partial^\gamma h_{\beta\nu\rho}^C + \frac{3}{4} h_a^{\alpha\beta} \square \partial_\gamma h_\alpha^B \partial^\gamma h_\beta^C - \frac{3}{4} h_a^{\alpha\beta} \square \partial_\gamma h_{\alpha\mu\nu}^B \partial^\gamma h_\beta^{C\mu\nu} \\
& - \frac{3}{4} h_a^{\alpha\beta} \partial^{3\gamma\mu\nu} h_{\alpha\mu\nu}^B \partial_\gamma h_\beta^C + \frac{3}{4} h_a^{\alpha\beta} \partial^{3\gamma\mu\nu} h_\alpha^B \partial_\gamma h_{\beta\mu\nu}^C + \frac{3}{4} h_a \square \partial^\alpha h^{B\beta} \partial_\beta h_\alpha^C \\
& - \frac{1}{2} h_a \partial^{3\mu\nu\rho} h^{B\alpha} \partial_\alpha h_{\mu\nu\rho}^C - \frac{3}{4} h_a \partial_{\mu\nu\rho}^3 h^{B\alpha\mu\nu} \partial_\alpha h^{C\rho} - \frac{1}{2} h_a \square \partial^\rho h^{B\alpha\mu\nu} \partial_\alpha h_{\mu\nu\rho}^C \\
& + h_a \partial_{\mu\rho\sigma}^3 h^{B\alpha\mu\nu} \partial_\alpha h_\nu^{C\rho\sigma} - \frac{3}{4} h_a^{\alpha\beta} \square \partial_\mu h_{\gamma\alpha\beta}^B \partial^\gamma h^{C\mu} + \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\nu\rho}^3 h_{\gamma\alpha\beta}^B \partial^\gamma h^{C\mu\nu\rho} \\
& - \frac{3}{4} h_a^{\alpha\beta} \partial_{\mu\alpha\beta}^3 h_\gamma^B \partial^\gamma h^{C\mu} + \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\alpha\beta}^3 h_{\gamma\nu\rho}^B \partial^\gamma h^{C\mu\nu\rho} - h_a^{\alpha\beta} \square \partial^\mu h_\gamma^B \partial_\gamma h_{\mu\alpha\beta}^C \\
& + h_a^{\alpha\beta} \partial^{3\mu\nu\rho} h_{\gamma\nu\rho}^B \partial^\gamma h_{C\mu\alpha\beta} - h_a^{\alpha\beta} \partial_{\alpha\mu\nu}^3 h_{\beta\gamma\rho}^B \partial^\gamma h^{C\mu\nu\rho} + \frac{3}{2} h_a^{\alpha\beta} \partial_{\alpha\mu\nu}^3 h_\beta^{B\gamma\mu} \partial_\gamma h^{C\nu} \\
& - \frac{3}{2} h_a^{\alpha\beta} \partial^{3\mu\nu\rho} h_{\alpha\gamma\rho}^B \partial^\gamma h_{\beta\mu\nu}^C + \frac{3}{2} h_a^{\alpha\beta} \square \partial^\mu h_\alpha^{B\gamma\nu} \partial_\gamma h_{\beta\mu\nu}^C + \frac{3}{2} h_a^{\alpha\beta} \partial_{\alpha\mu\nu}^3 h_\gamma^B \partial^\gamma h_{\beta}^{C\mu\nu} \\
& - \frac{3}{2} h_a^{\alpha\beta} \partial_{\alpha\mu\nu} h^{B\gamma\mu\rho} \partial_\gamma h_{\beta\rho}^{C\nu} + \frac{3}{4} h_a^{\alpha\beta} \square \partial_\mu h^{B\mu} \partial^\nu h_{\nu\alpha\beta}^C - \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\nu\rho}^3 h^{B\mu\nu\rho} \partial^\sigma h_{\sigma\alpha\beta}^C \\
& - \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu\nu}^3 h^{B\mu} \partial_\gamma h_\beta^{C\gamma\nu} - \frac{1}{2} h_a^{\alpha\beta} \square \partial^\mu h_\alpha^B \partial^\gamma h_{\beta\gamma\mu}^C + \frac{1}{2} h_a^{\alpha\beta} \partial^{3\mu\nu\rho} h_{\alpha\nu\rho}^B \partial^\gamma h_{\beta\gamma\mu}^C \\
& - \frac{1}{4} h_a \square h^{B\alpha} \square h_\alpha^C + \frac{1}{8} h_a \square h^{B\alpha\beta\gamma} \square h_{\alpha\beta\gamma}^C + \frac{1}{8} h_a \square h^{B\alpha} \partial_{\alpha\beta}^2 h^{C\beta} \\
& - \frac{3}{4} h_a \square h^{B\rho\sigma\alpha} \partial_{\alpha\beta}^2 h_{\rho\sigma}^{C\beta} + \frac{3}{4} h_a \square h^{B\alpha\beta\mu} \partial_{\alpha\beta}^2 h_\mu^C + \frac{1}{2} h_a \square h_\mu^B \partial_{\alpha\beta}^2 h^{C\alpha\beta\mu} \\
& + \frac{1}{4} h_a \partial^{2\alpha\beta} h^{B\mu} \partial_{\alpha\beta}^2 h_\mu^C + \frac{3}{8} h_a \partial^{2\alpha\beta} h^{B\mu\nu\rho} \partial_{\alpha\beta}^2 h_{\mu\nu\rho}^C + \frac{1}{8} h_a \partial^{2\alpha\mu} h_\mu^B \partial_{\alpha\nu}^2 h^{C\nu} \\
& - \frac{1}{8} h_a \partial^{2\alpha\mu} h_{\mu\rho\sigma}^B \partial_{\alpha\nu}^2 h^{C\nu\rho\sigma} + \frac{7}{8} h_a \partial^{2\alpha\mu} h^{B\nu} \partial_{\alpha\nu}^2 h_\mu^C - \frac{5}{8} h_a \partial^{2\alpha\mu} h^{B\nu\rho\sigma} \partial_{\alpha\nu}^2 h_{\mu\rho\sigma}^C \\
& - h_a \partial^{2\alpha\mu} h_{\mu\nu\rho}^B \partial_\alpha^2 h_\nu^{C\rho} - \frac{1}{4} h_a \partial^{2\alpha\beta} h_{\alpha\beta\gamma}^B \partial_{\mu\nu}^2 h^{C\mu\nu\gamma} - \frac{1}{8} h_a \partial^{2\alpha\beta} h_{\alpha\beta\gamma}^B \partial^{2\gamma\mu} h_\mu^C \\
& + \frac{1}{4} h_a \partial^{2\alpha\beta} h^{B\mu\nu\gamma} \partial_{\mu\nu}^2 h_{\alpha\beta\gamma}^C - \frac{5}{4} h_a \partial^{2\alpha\beta} h_\mu^B \partial^{2\eta\mu} h_{\alpha\beta\gamma}^C + \frac{5}{4} h_a \partial^{2\alpha\beta} h_{\alpha\mu\gamma}^B \partial^{2\mu\nu} h_{\beta\nu}^{C\gamma} \\
& + \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h^{B\gamma} \square h_\gamma^C - \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h^{B\mu\nu\rho} \square h_{\mu\nu\rho}^C - \frac{3}{4} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h^{B\mu\nu\rho} \partial_{\mu\nu}^2 h_\rho^C \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h_\rho^B \partial_{\mu\rho}^2 h^{C\mu\nu\rho} - \frac{1}{8} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h^{B\mu} \partial_{\mu\nu}^2 h^{C\nu} + \frac{3}{4} h_a^{\alpha\beta} \partial_{\alpha\beta}^2 h^{B\rho\sigma\mu} \partial_{\mu\nu}^2 h_{\rho\sigma}^{C\nu}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} h_a^{\alpha\beta} \square h_{\alpha\beta\gamma}^B \square h^{C\gamma} - \frac{1}{2} h_a^{\alpha\beta} \square h_{\alpha\beta\gamma}^B \partial_{\mu\nu}^2 h^{C\gamma\mu\nu} - \frac{3}{4} h_a^{\alpha\beta} \partial_{\mu\nu}^2 h_{\alpha\beta\gamma}^B \square h^{C\gamma\mu\nu} \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\nu}^2 h_{\alpha\beta\gamma}^B \partial^{2\mu\nu} h^{C\gamma} + h_a^{\alpha\beta} \partial_{\mu\nu}^2 h_{\alpha\beta\gamma}^B \partial_{\rho}^{2\mu} h^{C\gamma\nu\rho} + \frac{3}{4} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\beta\mu}^B \partial_{\nu\tau}^2 h^{C\tau} \\
& + \frac{3}{4} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\beta\mu}^B \square h_{\nu}^C - \frac{3}{4} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\beta\mu}^B \partial^{2\rho\sigma} h_{\nu\rho\sigma}^C - \frac{7}{4} h_a^{\alpha\beta} \partial_{\nu\tau}^2 h_{\alpha\beta\mu}^B \partial^{2\mu\nu} h^{C\tau} \\
& - \frac{1}{8} h_a^{\alpha\beta} \square h_{\alpha\beta\mu}^B \partial^{2\mu\nu} h_{\nu}^C + \frac{5}{4} h_a^{\alpha\beta} \partial^{2\rho\sigma} h_{\alpha\beta\mu}^B \partial^{2\mu\nu} h_{\nu\rho\sigma}^C - \frac{5}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^B \square h^{C\mu} \\
& + \frac{5}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta\nu\rho}^B \partial^{2\nu\rho} h^{C\mu} + \frac{3}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta\nu\rho}^B \square h^{C\mu\nu\rho} + \frac{5}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^B \partial_{\nu\rho}^2 h^{C\mu\nu\rho} \\
& - \frac{5}{2} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta\nu\rho}^B \partial_{\tau}^{2\rho} h^{C\nu\rho\tau} + \frac{1}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^{B\mu\nu} \partial_{\nu\rho}^2 h^{C\rho} - h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^{B\mu\nu} \square h_{\nu}^C \\
& + h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^{B\mu\nu} \partial^{2\rho\sigma} h_{\nu\rho\sigma}^C + h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta\nu\rho}^B \partial^{2\mu\nu} h^{C\rho} - \frac{11}{8} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^B \partial^{2\mu\nu} h_{\nu}^C \\
& + \frac{1}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\beta}^{B\rho\sigma} \partial^{2\mu\nu} h_{\nu\rho\sigma}^C + \frac{5}{8} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\mu} \square h_{\beta}^C - \frac{1}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\mu\nu\rho} \partial_{\nu\rho}^2 h_{\beta}^C \\
& + \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\mu\nu\rho} \square h_{\beta\nu\rho}^C - \frac{3}{4} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\mu} \partial^{2\nu\rho} h_{\beta\nu\rho}^C - h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\mu\nu\rho} \partial_{\nu\tau}^2 h_{\beta\rho}^{C\tau} \\
& + h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\tau} \partial_{\nu\tau}^2 h_{\beta}^{C\mu\nu} - h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\nu}^B \square h_{\beta}^{C\mu\nu} + h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h^{B\tau} \partial^{2\mu\nu} h_{\beta\nu\tau}^C \\
& + \frac{1}{2} h_a^{\alpha\beta} \partial_{\mu\alpha}^2 h_{\nu}^B \partial^{2\mu\nu} h_{\beta}^C - \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\nu} \partial_{\beta}^{2\mu} h_{\nu}^C - \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\nu\rho\sigma} \partial_{\beta}^{2\mu} h_{\nu\rho\sigma}^C \\
& + \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\mu} \partial_{\beta\nu}^2 h^{C\nu} - \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\mu\rho\sigma} \partial_{\beta}^{2\nu} h_{\nu\rho\sigma}^C - \frac{1}{2} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\nu} \partial_{\beta\nu}^2 h^{C\mu} \\
& + \frac{1}{4} h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\nu\rho\sigma} \partial_{\beta\nu}^2 h_{\rho\sigma}^{C\mu} + h_a^{\alpha\beta} \partial_{\alpha\mu}^2 h^{B\mu\nu\rho} \partial_{\beta\nu}^2 h_{\rho}^C + \frac{1}{4} h_a^{\alpha\beta} \square h_{\alpha}^B \square h_{\beta}^C \\
& - \frac{1}{4} h_a^{\alpha\beta} \square h_{\alpha}^{B\mu\nu} \square h_{\beta\mu\nu}^C + \frac{1}{4} h_a^{\alpha\beta} \square h_{\alpha\mu\nu}^B \partial^{2\mu\nu} h_{\beta}^C - \frac{3}{4} h_a^{\alpha\beta} \square h_{\alpha}^B \partial^{2\mu\nu} h_{\beta\mu\nu}^C \\
& + h_a^{\alpha\beta} \square h_{\alpha\mu}^B \partial_{\nu\rho}^2 h_{\beta}^{C\mu\nu} + \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\mu\nu}^B \partial^{2\rho\sigma} h_{\beta\rho\sigma}^C - \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\rho\sigma}^B \partial^{2\rho\sigma} h_{\beta\mu\nu}^C \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha\mu\rho}^B \partial^{2\rho\sigma} h_{\beta\nu\sigma}^C + \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha}^B \partial_{\mu\nu}^2 h_{\beta}^C - \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\nu} h_{\alpha}^{B\rho\sigma} \partial_{\mu\nu}^2 h_{\beta\rho\sigma}^C \\
& - \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\rho} h_{\alpha\rho\sigma}^B \partial_{\mu}^{2\sigma} h_{\beta}^C - \frac{1}{2} h_a^{\alpha\beta} \partial^{2\mu\rho} h_{\alpha\rho\tau}^B \partial_{\mu\sigma}^2 h_{\beta}^{C\sigma\tau} + h_a^{\alpha\beta} \partial^{2\mu\rho} h_{\alpha}^{B\sigma\tau} \partial_{\mu\sigma}^2 h_{\beta\rho\tau}^C.
\end{aligned}$$

## References

- [1] C. Fronsdal, Phys. Rev. D **18** (1978) 3624.
- [2] M. A. Vasiliev, Phys. Lett. B **567** (2003) 139 [[hep-th/0304049](#)];  
Fortsch. Phys. **52** (2004) 702 [[hep-th/0401177](#)];  
Comptes Rendus Physique **5** (2004) 1101 [[hep-th/0409260](#)].
- [3] B. de Wit and D. Z. Freedman, Phys. Rev. D **21** (1980) 358.
- [4] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Nucl. Phys. B **597** (2001) 127 [[hep-th/0007220](#)].

- [5] N. Boulanger and L. Gualtieri, *Class. Quant. Grav.* **18** (2001) 1485 [[hep-th/0012003](#)].
- [6] X. Bekaert, N. Boulanger and S. Cnockaert, *JHEP* **0601** (2006) 052 [[hep-th/0508048](#)].
- [7] N. Boulanger, S. Leclercq and S. Cnockaert, *Phys. Rev. D* **73** (2006) 065019 [[hep-th/0509118](#)].
- [8] G. Barnich and M. Henneaux, *Phys. Lett. B* **311** (1993) 123 [[hep-th/9304057](#)].
- [9] M. Henneaux, *Contemp. Math.* **219** (1998) 93 [[hep-th/9712226](#)].
- [10] N. Boulanger, *Fortsch. Phys.* **50** (2002) 858 [[hep-th/0111216](#)].
- [11] X. Bekaert, N. Boulanger, S. Cnockaert and S. Leclercq, *Fortsch. Phys.* **54** (2006) 282 [[hep-th/0602092](#)].
- [12] R. R. Metsaev, *Cubic interaction vertices of massive and massless higher spin fields*, [hep-th/0512342](#).
- [13] I. L. Buchbinder, A. Fotopoulos, A. C. Petkou and M. Tsulaia, *Constructing the cubic interaction vertex of higher spin gauge fields*, [hep-th/0609082](#).
- [14] F. A. Berends, G. J. H. Burgers and H. van Dam, *Nucl. Phys. B* **260** (1985) 295.
- [15] F. A. Berends, G. J. H. Burgers and H. van Dam, *Nucl. Phys. B* **271** (1986) 429.
- [16] C. Aragone and H. La Roche, *Nuovo Cim. A* **72** (1982) 149.
- [17] S. Deser and Z. Yang, *Class. Quant. Grav.* **7** (1990) 1491.
- [18] S. Weinberg, *Phys. Rev.* **138** (1965) B988.
- [19] T. Damour and S. Deser, *Class. Quant. Grav.* **4** (1987) L95.
- [20] F. A. Berends, G. J. H. Burgers and H. Van Dam, *Z. Phys. C* **24** (1984) 247.
- [21] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong  $Sp(2, \mathbb{R})$  condition*, [[hep-th/0501156](#)];  
X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear Higher Spin Theories in Various Dimensions*, [[hep-th/0503128](#)].  
Both works published in the proceedings of the First Solvay Workshop on Higher Spin Gauge Theories (Brussels, May 2004).

- [22] E. S. Fradkin and M. A. Vasiliev, Phys. Lett. B **189** (1987) 89;  
Nucl. Phys. B **291** (1987) 141;  
M. A. Vasiliev, Nucl. Phys. B **616** (2001) 106 [Erratum-ibid. B **652** (2003) 407]  
[hep-th/0106200];  
K. B. Alkalaev and M. A. Vasiliev, Nucl. Phys. B **655** (2003) 57 [hep-th/0206068].
- [23] X. Bekaert and N. Boulanger, Phys. Lett. B **561** (2003) 183 [hep-th/0301243];  
*Mixed symmetry gauge fields in a flat background*, in the proceedings of the International  
Seminar on Supersymmetries and Quantum Symmetries “SQS 03” (Dubna, July 2003)  
[hep-th/0310209].
- [24] D. Francia and A. Sagnotti, Phys. Lett. B **543** (2002) 303 [hep-th/0207002];  
Class. Quant. Grav. **20** (2003) S473 [hep-th/0212185].
- [25] T. Damour and S. Deser, Annales Poincaré Phys. Théor. **47** (1987) 277.
- [26] M. Dubois-Violette and M. Henneaux, Lett. Math. Phys. **49** (1999) 245 [math.qa/9907135];  
Commun. Math. Phys. **226** (2002) 393 [math.qa/0110088].
- [27] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of  $GL(D,R)$ .  
II: Quadratic actions*, To be published in Commun. Math. Phys. [hep-th/0606198].
- [28] X. Bekaert and N. Boulanger, Nucl. Phys. B **722** (2005) 225 [hep-th/0505068].
- [29] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174** (1995) 57  
[hep-th/9405109].
- [30] G. Barnich, N. Bouatta and M. Grigoriev, JHEP **0510** (2005) 010 [hep-th/0507138].
- [31] N. Bouatta, *Approche BRST des systèmes invariants de jauge à spins élevés*, Master Degree  
Thesis defended at the Free University of Brussels (Academic year 2003-2004).
- [32] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174** (1995) 93  
[hep-th/9405194].
- [33] J. A. M. Vermaseren, “New features of FORM,” math-ph/0010025.